

Collateral Constraints in a Monetary Economy^{*†}

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This paper studies the role of collateral constraints in transforming small monetary shocks into large persistent output fluctuations. We do this by introducing money in the heterogeneous-agent real economy of Kiyotaki and Moore (1997). Money enters in a cash-in-advance constraint and money supply is managed via open-market operations. We find that a monetary shock generates persistent movements in aggregate output, the amplitude of which depends upon whether or not debt contracts are indexed. If only nominal contracts are traded, money shocks can trigger large output fluctuations. In this case a money expansion triggers a boom, while money contractions generate recessions. In contrast, if contracts are indexed amplification is not only smaller, but it can generate the reverse results. When the possibility of default and renegotiation is considered, the model can generate asymmetric business cycles with recessions milder than booms. Finally, monetary shocks generate a highly persistent dampening cycle rather than a smoothly declining deviation.

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1. Introduction

The extent and mechanism through which monetary policy affects real economic activity over business cycles has been a long-standing question in macroeconomics. Different mechanisms that explain the propagation of money shocks have been proposed. These include sticky prices, wage contracting, monetary misperceptions, and limited participation.³ Another mechanism that has received special attention in recent years is credit-market imperfections. In particular, the agency-cost model of Bernanke and Gertler (1989) has been extended to monetary environments in order to analyze how fluctuations in borrowers' net worth can contribute to the amplification and persistence of exogenous money shocks to the economy: e.g., Fuerst (1995), Bernanke, Gertler and Gilchrist (1999), and Carlstrom and Fuerst (2000).⁴

In contrast with these agency-cost models, little attention has been devoted to analyzing *monetary* economies in which agents face endogenous credit limits determined by the value of collateralized assets. The environment we have in mind is one in which lenders cannot force borrowers to repay their debts unless debts are secured. Using *real*-economy models, Kiyotaki and Moore (1997), Kiyotaki (1998) and Kocherlakota (2000) among others, have shown that collateral constraints may be a powerful mech-

³See Cooley and Hansen (1998) for an illustration of the role of monetary shocks in the equilibrium business cycle theory.

⁴Credit-market imperfections in these models emerge from asymmetric information and costly state verification. In this framework, entrepreneurs borrow to pay the amount of the factor bill that is not covered by their net worth. Lenders must pay a monitoring cost in order to observe the entrepreneur's project outcome. If an entrepreneur has little net worth invested in the project, monitoring costs increase because there is larger divergence between the interests of the entrepreneur and the lender, and so the premium for external financing is larger. With procyclical net worth, periods of low output are associated with higher monitoring costs and a higher external finance premium. This mechanism amplifies the effects of external shocks on production and investment.

anism of amplification and persistence of real shocks.⁵ These papers show that when debts must be fully secured by collateral (say land) and the collateral is also an input in production, then a small shock to the economy can be significantly amplified. For instance, a small negative shock that reduces the net worth of credit-constrained firms forces them to curtail their investment in land. Land prices and output fall because credit-constrained firms are by nature more productive in the use of land. The fall in the value of the collateral further reduces the debt capacity of constrained firms, causing additional declines in investment, land prices, and output.

This paper studies the potential role of collateral constraints as a transmission mechanism of monetary shocks. We do this by introducing a cash-in-advance constraint for consumption and investment in the real-economy model of Kiyotaki and Moore (1997). We exploit the simplicity of this framework to study monetary injections carried out via open-market operations, as opposed to the less-realistic but simpler helicopter drops employed by many monetary models. Due to the presence of credit-market imperfections, the exact path of the money supply is important to determine the real effects of open-market operations. We choose a parsimonious type of monetary paths that avoid changes in long-run inflation and fiscal variables. Thus, current monetary expansions must be offset by future monetary contractions to avoid changes in inflation or unstable government-bond paths. As in real-economy models, the price of the collateral plays a central role in generating large and persistent effects of exogenous shocks. Moreover, the response of the nominal interest rate becomes also critical in determining the effects of shocks.⁶

The main finding of this paper is that a monetary shock can generate persistent

⁵Scheinkman and Weiss (1986) also study the effects of *borrowing* constraints in the presence of uninsurable risk. They simulate a lump-sum monetary injection that changes the distribution of assets across agents.

⁶In Kiyotaki and Moore (1997) and Kocherlakota (2000) the equilibrium interest rate is constant.

movements in aggregate output, the amplitude of which depends on whether or not debt contracts are indexed. In particular, if indexed contracts cannot be written, then unanticipated money shocks can trigger large output fluctuations. In this case an unanticipated money expansion triggers an economic boom, while unanticipated money contractions generate depressions. The basic mechanism at work is the Fisher effect, according to which unexpected debt-deflation (even if small) redistributes resources from borrowers to lenders. In our model, due to the existence of collateral constraints, this Fisher effect can amplify output fluctuations significantly. In contrast, if contracts are indexed, output amplification is not only smaller, but it is possible for money expansions to generate downturns in output. This occurs because even though indexed contracts prevent any redistribution of resources from lenders to borrowers, the inflationary tax reduces borrowers' net worth.

Unanticipated shocks may induce default and renegotiation if the value of debts exceed the collateral value *expost*. The possibility of renegotiation can generate asymmetric business cycles as default may occur during depressions but not during booms. We find that if contracts are non-indexed, an unanticipated monetary contraction reduces the value of the collateral and therefore may induce default and renegotiation, depending on the timing of the shock. Renegotiation benefits borrowers and prevents a larger output downturn; i.e., renegotiation can substantially reduce the responses of output to shocks.

A third property of the model is that monetary shocks trigger highly persistent dampening *cycles* rather than smoothly declining deviations. This occurs due to the interplay between the cash-in-advance and collateral constraints. In particular, investment in the cash-in-advance constraint is analogous to a “time to build” or an “adjustment cost” restriction. As a result, the full impact of a shock that increases net worth is delayed because with a binding cash-in-advance constraint, collateral

can only be accumulated gradually. The cyclical dynamics of the model is consistent with the hump-shaped pattern of output response to monetary shocks that has been observed in the data.⁷

Finally, the model also generates endogenous limited participation in the government-bonds market due to the fact that in equilibrium, collateral constraints are binding only for a set of agents. This implies that only unconstrained agents hold government bonds and participate in open-market operations. In this context, the propagation of the money shock is nontrivial because agents differ not only in whether or not they are credit constrained, but also in their productivity.

The remainder of the paper is organized as follows. Section 2 presents the model and characterizes the steady state. In Section 3 we discuss the dynamics of the model in response to a monetary shock. The dynamic structure of the model can be summarized by a nonhomogeneous second-order difference equation in the distribution of capital across agents. We parameterize the model and provide a numerical illustration of the dynamics in Section 4. Finally, Section 5 concludes. Technical details omitted in the text are presented in the Appendix.

2. The model

The model for this heterogeneous-agent economy is an extension of the framework of Kiyotaki and Moore (1997). We keep the main features of their model and introduce money using a cash-in-advance (CIA) constraint. There are two goods in this economy: a durable asset (capital), and a nondurable commodity (output). We focus on the effects of monetary shocks on the distribution of capital across agents and abstract from capital accumulation. Capital is available in an aggregate fixed amount \bar{K} .

There are two types of private agents in this economy. They are both risk neutral,

⁷See Bernanke, Gertler and Gilchrist (1999), and Carlstrom and Fuerst (2000).

but operate different technologies and have distinct discount factors. As will become clear below, around the steady state the more patient agents become lenders, while the impatient agents become borrowers. To abbreviate, let us refer to the two types of agents as borrowers and lenders. Both types of agents face a cash-in-advance (CIA) constraint and a collateral constraint. Finally, the government in this economy has the only role of controlling money supply through open-market operations.

Events in this model occur as follows. Production takes place overnight. A household can default its debt by escaping with the production and money balances overnight, and leaving their installed capital behind. Early in the morning, lenders seize the capital of the defaulting households. No default occurs during the day as movers can be easily detected, and their output, capital, and money balances can be seized. We assume that there are different “types of capital,” so that lenders cannot directly use seized capital, but they have to sell it and then buy their correct capital “type.” As in other CIA models, we also assume that households value the different “types” of output produced by other households.

Assume that there are two members per household who carry out different activities: “shoppers” and “producers”. Households enter each period with money balances stored from the previous period; output that was produced overnight; and capital holdings, if any. During the day, all markets are opened simultaneously. Shoppers leave early in the morning with their money balances to make transactions in both the capital and goods markets. They can buy or sell capital, and buy goods. We assume that each period shoppers can only make one transaction in the capital market; i.e., they can *either buy or sell* capital, but they do not have “enough time” to both buy and sell. One can think that buying and selling capital goods takes more time, or it is “more involved” than buying and selling consumption goods. This assumption, together with the existence of different types of capital, avoids default becoming a

cheaper way of acquiring capital.⁸

Finally, “producers” stay at home selling the production, and making transactions in the money and bonds markets. In particular, producers pay outstanding debts using the revenues from output sales made during the day. Financial transactions must satisfy a standard budget constraint for the producer, as well as a collateral constraint. Monetary interventions occur via open-market operations. In particular, the government can only expand and contract the money supply by buying or selling bonds. These open-market operations take place after shoppers have left for the day.

The assumed market arrangement means that cash is required for trading capital and consumption goods but not for financial transactions. Producers can pay outstanding debts with cash generated during the day, or equivalently, they could pay directly with goods. The important point is that it is not necessary to accumulate cash in advance in order to carry out financial transactions. We could alternatively restrict financial transactions to be cash-based as in, e.g., Freeman (1996), and Aiyagari et al. (1998). In this case, monetary shocks could have a direct impact on financial markets. Our research strategy is to keep separate the financial and monetary elements of the model in order to focus on the effects of nominal versus indexed financial contracts (i.e., the Fisher effect).

2.1. Lenders

The mass of lenders in the economy is n . Lenders differ from borrowers in their production technology and preferences. Lenders use a strictly concave technology, and they are more patient than borrowers. Their production function is given by

⁸In other words, by restricting the number of transactions in the capital market we prevent lenders from selling seized capital and then buying new capital immediately after. This would make default a cheaper way to acquire capital because no cash accumulated from the previous period would be required.

$y_t = G(k'_{t-1})$, where $G' > 0$, $G'' < 0$, $G'(0) = \infty$, and k'_{t-1} is their capital stock at the end of last period. Lenders choose sequences of consumption $\{x'_t\}$, capital holdings $\{k'_t\}$, nominal money balances $\{m_t^{dj}\}$, bonds holdings $\{b'_t\}$, and government-bonds purchases $\{h'_t\}$, to solve the following problem for given sequences of output prices $\{p_t\}$, nominal interest rates $\{R_t\}$, and nominal capital prices $\{q_t^n\}$.

$$\max \sum_{t=0}^{\infty} \beta'^t x'_t,$$

subject to

$$(1) \quad q_t^n (k'_t - k'_{t-1}) + p_t x'_t = m_{t-1}^{dj},$$

$$(2) \quad m_t^{dj} + R_t b'_{t-1} + h'_t = p_t G(k'_{t-1}) + b'_t + R_t h'_{t-1},$$

where the prime denotes a lender's decision variable.⁹ We define the nominal rate R_t as the interest rate paid at t on loans made at $t-1$. Equation (1) is the CIA constraint. Money is required for both consumption $p_t x'_t$ and investment $q_t^n (k'_t - k'_{t-1})$. Equation (2) is the budget constraint. The revenues collected through output sales $p_t G(k'_{t-1})$, new bonds issued b'_t , and the proceeds from government-bond holdings $R_t h'_{t-1}$ must be enough to accumulate new money balances m_t^{dj} , pay outstanding debt obligations $R_t b'_{t-1}$, and purchase government bonds h'_t .

Let $\beta'^t \Omega_t$ be the Lagrange multiplier associated to the CIA constraint and $\beta'^t \Lambda_t$ the one for the budget constraint. Then, the first-order optimality conditions for the problem above are given by¹⁰

$$x'_t : 1 = \Omega_t p_t,$$

$$m_t^{dj} : \Lambda_t = \beta' \Omega_{t+1},$$

⁹In order to avoid introducing more notation, we have assumed that the constraints are binding, as it will be the case in equilibrium.

¹⁰These are the first-order conditions for interior solutions; assumptions below guarantee their attainment.

$$b'_t, h'_t : \Lambda_t = \beta' R_{t+1} \Lambda_{t+1},$$

$$k'_t : q'_t \Omega_t - \beta' q'_{t+1} \Omega_{t+1} = \beta' \Lambda_{t+1} p_{t+1} G'(k'_t).$$

Let the real price of capital be $q_t \equiv q'_t/p_t$. From the optimality conditions above is easy to obtain expressions for the nominal interest rate and the users cost of capital for the lenders $u' \equiv q_t - \beta' q_{t+1}$,

$$(3) \quad R_t = \frac{1 + \pi_t}{\beta'},$$

$$(4) \quad u'_t = \frac{\beta'^2}{1 + \pi_{t+1}} G'(k'_t),$$

where $\pi_t = \frac{p_{t+1} - p_t}{p_t}$ is the inflation between t and $t+1$. Equation (4) states that lenders equate their users cost of capital with the present value of its marginal product. Since in equilibrium these agents are not credit constrained, the users cost is simply the difference between the cost of buying capital today q_t and the discounted value of selling capital tomorrow $\beta' q_{t+1}$. Notice that the lenders' users cost is not affected by inflation since the proceeds of selling the capital can be consumed or invested immediately, without requiring a previous accumulation of cash. In contrast, the marginal product of capital has to be discounted by $\frac{\beta'^2}{1 + \pi_{t+1}}$ because output must be exchanged for money before it can be consumed. This means that the investor has to wait two periods and pay the inflation tax before he can consume the returns of the investment.

2.2. Borrowers

The measure of borrowers is normalized to one. Their technology is given by the production function $y_t = (a + c)k_{t-1}$. They choose sequences of consumption $\{x_t\}$, capital holdings $\{k_t\}$, nominal money balances $\{m_t^d\}$, private-issued bonds $\{b_t\}$, and government-bonds purchases $\{h_t\}$ to solve the following problem for given sequences

of output prices, nominal capital prices, nominal interest rates, and government-bonds nominal rates

$$\max \sum_{t=0}^{\infty} \beta^t x_t,$$

subject to

$$(5) \quad q_t^n (k_t - k_{t-1}) + p_t x_t = m_{t-1}^d,$$

$$(6) \quad m_t^d + R_t b_{t-1} + h_t = (a + c) p_t k_{t-1} + b_t + R_t h_{t-1},$$

$$(7) \quad R_{t+1} b_t \leq q_{t+1}^n k_t.$$

In addition to the CIA and budget constraints, borrowers face a collateral constraint, given by equation (7). Borrowing can only take place up to the point at which the principal plus interest $R_{t+1} b_t$ is secured by the market value of the capital owned by the household $q_{t+1}^n k_t$. Notice that money cannot be used as collateral.¹¹ Lenders also face a collateral constraint but we did not write it explicitly. Around the steady state this constraint is not binding under the following assumption¹²

Assumption 1. $\beta' > \beta$.

It is also assumed that only the fraction a of the output is tradable between borrowers and lenders. The fraction c can be traded only among borrowers, and it can be interpreted as a subsistence minimum consumption. We refer to this fraction

¹¹Money is not collateral because the borrower can hide it, or escape with it “overnight”, while it is not possible for the borrower to either hide or escape with installed capital. The fact that money is not collateral reduces the debt-to-asset ratio in our model, and weakens the leverage effect. This is analogous to the role that non-collateralizable trees play in Section III of Kiyotaki and Moore (1997).

We thank an anonymous referee for pointing this out to us.

¹²As in Kiyotaki and Moore (1997), under Assumption 1 there is no steady-state equilibrium in which the collateral constraint for *borrowers* is not binding. This is not true if $\beta \geq \beta'$.

as nontradable output. The purpose of the assumption is to avoid situations in which borrowers continuously postpone consumption.¹³

In Appendix A we prove that around a zero-inflation steady state the borrower's optimal plan is to consume only the nontradable fraction of output, i.e., $x_t = ck_{t-1}$, to borrow up to the limit imposed by the collateral constraint, and to invest all remaining resources. This implies that borrowers do not purchase government bonds, i.e., $h_t = 0$, and that the CIA constraint is binding. Notice that since borrowers do not hold government bonds while lenders do, this model generates endogenous limited participation in this market. These results hold under the following additional assumption

Assumption 2.

$$\frac{c}{a} > \frac{(1-\beta)(2-\beta-\beta')}{\beta^2(1-\beta')}.$$

This condition is easy to satisfy if the discount factors are close to 1.¹⁴ We can use equations (5), (6) and (7) to obtain

$$(8) \quad k_t = \frac{1}{u_t} \left[(a + q_t)k_{t-1} + \frac{m_{t-1}^d}{p_t} - R_t \frac{b_{t-1}}{p_t} - \frac{m_t^d}{p_t} \right].$$

Borrowers' net worth consists of their assets net of debt repayments. In the equation above real net worth corresponds to the term $(a + q_t)k_{t-1} + \frac{m_{t-1}^d}{p_t} - R_t \frac{b_{t-1}}{p_t}$; that is, the value of tradable output and capital held from the previous period, the

¹³Kiyotaki and Moore (1997) introduce a similar assumption. As will be explained below, due to the linearity of preferences, borrowers would like to continuously postpone consumption in exchange for investment. This is avoided by introducing a nontradable fraction of output, which we think of as subsistence minimum consumption. Notice that money is required to buy nontradable output because this type of output can be traded among borrowers. One can think that households can only produce say fruit of a particular color, but they value fruits of all colors.

¹⁴In this case, $\frac{(2-\beta-\beta')}{(1-\beta')}$ is some constant near to 2, and $\frac{(1-\beta)}{\beta^2}$ is close to zero. Further, in the proposed equilibrium $\frac{c}{a}$ is the ratio between the marginal propensity to consume and the marginal propensity to save for borrowers, which can be assumed to be bounded away from zero.

real money balances brought from the previous period, all net of debt repayments.

Finally, the users cost of capital for borrowers u_t , is given by

$$(9) \quad u_t \equiv q_t - \frac{1 + \pi_t}{R_{t+1}} q_{t+1}.$$

where borrowers discount the future value of the capital at the nominal interest rate R_{t+1} because, as will become clear below, in equilibrium borrowers need to borrow in order to buy capital.

2.3. Government

The government controls the money supply in this economy through open-market operations (OMOs), which take place in the bonds market. Let H_t^s be the nominal supply of government-issued bonds. The stock of money supply, M_t^s , in this economy is given by

$$(10) \quad M_t^s = M_{t-1}^s - H_t^s + R_t H_{t-1}^s,$$

where

$$(11) \quad H_t^s = \tau H_{t-1}^s, \quad \tau \geq 0,$$

so that at time t the government withdraws an amount τH_{t-1}^s of money and injects $R_t H_{t-1}^s$ back into the economy. There are two comments in order. First, we choose a simple law of motion for government bonds H_t^s . This simplicity is convenient for the purpose of analyzing the effects of a money shock. Notice that following this shock, unless $\tau < 1$ for all t , government bonds may exhibit an explosive path. To avoid this, any money expansion through OMOs must be eventually followed by a “policy reversal” or “sterilization” that guarantees convergence back to the steady state. In particular, the size of τ determines the speed at which such a monetary contraction takes place. In the analysis below we consider τ very close to 1 in order to simulate

a very slow policy reversal. Since credit markets are imperfect in this economy, real effects of monetary shocks may depend on the path of government debt. Although we choose a parsimonious law of motion for H_t^s , we will discuss below the role of the size of τ in our results, as well as other paths for government debt.

Second, notice that we do not consider a rebate of the inflationary tax. Since some agents face corner solutions, such a rebate cannot be lump-sum in general. For example, simple helicopter drops redistribute wealth and affect agents' decisions. Since here we want to focus on the effects of the "pure monetary shock", we do not include any rebates in the model. Tax rebates in fact may reinforce the results of the paper.¹⁵

2.4. Aggregate resource constraints

Let $K_t, K'_t, B_t, B'_t, H_t, H'_t, M_t^d, M_t^{d'}$ be the aggregate variables corresponding to the lowercase individual variables. There are five markets in the model: consumption goods, capital, money, private bonds, and public bonds. By Walras' Law one needs only to consider four of them. The equilibrium conditions to clear the last four markets are

$$M_t^s = M_t^d + M_t^{d'} = m_t^d + nm_t^{d'},$$

$$B_t = b_t = -B'_t = -nb'_t,$$

$$\bar{K} = K_t + K'_t = k_t + nk'_t,$$

¹⁵The intuition for this result is simple. Suppose the economy starts off at the steady state and there is a money expansion. Assume that borrowers were to receive a money transfer that compensates them for the inflationary tax in an amount higher than their optimal consumption level. This may happen, for example, with helicopter drops. In this case, borrowers will buy capital with the extra resources, and next period output will increase. This reinforces our results because in general, as will be shown below, monetary expansions generate booms.

and since $H_t = 0$,

$$H_t' = nh_t' = H_t^s.$$

Using the market clearing conditions above along with equations (2) and (6) we obtain

$$(12) \quad M_t^s + H_t^s - R_t H_{t-1}^s \equiv M_{t-1}^s = p_t \left[(a + c)K_{t-1} + nG \left(\frac{\bar{K} - K_{t-1}}{n} \right) \right],$$

which is just the quantity equation. Notice that this equation implies that prices at time t are not affected by the money supply at time t , but rather by the money supply at $t - 1$. This is a consequence of the cash in advance constraint. Moreover, notice that p_t is completely determined by choices made at $t - 1$ since both K_{t-1} and M_{t-1}^s are determined at $t - 1$.

2.5. Steady state

Define a steady state in which all real variables are constant, and all nominal variables grow at the constant rate π , which is the steady-state growth rate of money supply. From equations (10) and (11) it follows that $\pi = \tau - 1$ if $\tau \geq 1$, and $\pi = 0$ if $\tau < 1$.

Next, it is easy to see that the steady-state users cost of capital for lenders and borrowers is the same: $u = u' = q(1 - \beta')$. Further, since under the proposed equilibrium the collateral constraint (7) binds for the borrowers, we can use R , u' and the budget constraint of these agents (6) to get: $u = a + c - \frac{M^d}{pK^*}$, where K^* is the borrowers' steady-state capital level. Next, using the CIA constraint (5) one obtains: $\frac{M^d}{p} = cK^*(1 + \pi)$; i.e., borrowers' real money balances exactly cover their consumption adjusted by inflation.

Combining the last two expressions we obtain: $u = a - \pi c$. Notice that if $\pi = 0$, we obtain the intuitive results that $\frac{M^d}{p} = cK^*$, and $u' = u = a$. This last equation means that in a steady state with no money growth, the users cost equals the tradable marginal product of capital.

Finally, using equation (4), we obtain an implicit solution K^*

$$(13) \quad G' \left(\frac{\bar{K} - K^*}{n} \right) = \frac{1 + \pi}{(\beta')^2} (a - \pi c).$$

Equation (13), along with Assumption 2, imply that in equilibrium borrowers have higher marginal products of capital than do lenders. The following proposition summarizes the main features of the steady state.

Proposition 1. *Under Assumptions 1 and 2,*

(i) *if $G'(\bar{K}/n) < \frac{1+\pi}{(\beta')^2} (a - \pi c)$ there exists a unique steady state;*

(ii) *$\frac{\partial K^*}{\partial \pi} \neq 0$ for $(1 + 2\pi)c \neq a$, so that inflation affects the steady-state output Y^* .*

Proof: The existence of a unique steady state level K^* is guaranteed from the properties of the production function $G(\cdot)$. It is easy to see that the left-hand side of equation (13) is continuous and strictly increasing in K , while the right-hand side is a constant. If $G'(\bar{K}/n) < \frac{1+\pi}{(\beta')^2} (a - \pi c)$ the left and right-hand side cross only once. Figure 1 illustrates the determination of the steady state. The second property follows easily ■

It is interesting to note that in the long run money is not superneutral, as indicated by Proposition 1, even though aggregate capital is constant. It is well known that money is not superneutral if investment enters in the CIA constraint, because inflation acts as a tax on capital accumulation (Abel, 1984). However, aggregate capital is constant in our model so that the standard result does not apply. The non-superneutrality arises because inflation acts as a tax for all agents, but at the margin it affects borrowers and lenders differently. Higher inflation decreases the marginal cost of investing for both types; i.e., it decreases the users cost of capital. For a given K^* , borrowers' net worths decrease with higher inflation because they must demand more money in order to sustain their consumption, cK^* . Further, since borrowers

are credit constrained, they are in a corner solution. In contrast, lenders have an interior solution and since u has decreased, their marginal benefit of investing must decrease, which can happen only if lenders' capital holdings $\bar{K} - K^*$ increase. Thus, money is not superneutral due to the asymmetric effect of inflation on constrained and unconstrained agents.¹⁶

3. Dynamics

To simplify the analysis, we only present the dynamics of the model around the steady state, and assume zero steady state inflation, $\pi = 0$. The solution for the case $\pi > 0$ is summarized in Appendix E. We also assume that β' is close to 1. This occurs, for example, if the length of the periods is small. This assumption allows us to obtain some sharp analytical results, but numerical simulations confirm that the main results hold even if β' is far from 1. Let $g_t \equiv \frac{M_t^s}{M_{t-1}^s}$ (i.e., g_t is one plus the growth rate of money supply) and $v_t \equiv \frac{p_{t+1}}{p_t}$ (i.e., v_t is one plus the inflation rate). Thus, in the steady state, $g = v = 1 + \pi$. In general, let $\hat{x}_t = \frac{x_t - x^*}{x^*}$ denote the rate of deviation of a variable x from its steady state value.

Assume that the economy starts off at the steady state, and that an unexpected increase in the growth rate of money $\varepsilon > 0$ occurs at $t = 0$, i.e., $\hat{g}_0 = \varepsilon$. Since the monetary expansion occurs through OMOs, H_0 decreases below its steady state

¹⁶If the borrowers' propensity to consume $\frac{c}{a+c}$ is larger than 0.5 then higher inflation reduces output, a result consistent with Abel (1984). However, if money is injected via helicopter drops rather than via OMOs, higher steady-state inflation may have the opposite effect, i.e. higher π implies larger K^* and larger Y^* . This occurs if borrowers receive a fraction of the transfer higher than their steady-state consumption share, $\alpha \equiv \frac{cK^*}{Y^*}$. In this case, borrowers are overcompensated for the inflationary tax and, as a result, they can afford to buy additional capital with the extra resources. In addition, inflation increases the marginal cost of investing u , but lenders are particularly hurt because they face an interior solution.

level ($\widehat{H}_0 < 0$). According to the law of motion for government bonds, $H_t^s = \tau H_{t-1}^s$, H_t gradually returns to zero to avoid changes in the long-term inflation rate. Thus, the money expansion at $t = 0$ is followed by a monetary contraction, i.e., by a “sterilization policy.” In particular, the size of $\tau < 1$ determines the speed at which such monetary contraction takes place.

Let the government-debt to money ratio be $d_t \equiv H_t^s/M_t^s$. Using the law of motion of money supply and bonds, one can obtain the following path of money growth¹⁷

$$\widehat{g}_0 = -\partial d_0,$$

and

$$\widehat{g}_t = -(R - \tau)\tau^{t-1}\widehat{g}_0.$$

Notice that this path is fully determined by the exogenous initial shock, and converges to zero at a rate determined by the size of τ . In particular, a larger τ implies a smoother sterilization of the initial monetary expansion. In what follows we will assume a τ very close to 1 in order to simulate a very slow monetary contraction.¹⁸

To complete the characterization of the dynamics of the model, we need to solve for the paths of \widehat{v}_t , \widehat{q}_t and \widehat{K}_t . Linearizing equation (12) yields

$$\widehat{v}_t = \widehat{g}_t - \rho \left(\widehat{K}_t - \widehat{K}_{t-1} \right),$$

¹⁷Here we compute the absolute deviations of the government-debt to money ratio ∂d_0 instead of the percentage deviations from the steady state because $d = 0$. The first expression follows from linearizing the stationary version of the equation $M_0^s = M_{-1}^s - H_0^s$. For $t \geq 1$, combine the law of motion for H_t^s and M_t^s , transform variables to render them stationary, and linearize to obtain $\widehat{g}_t = (R - \tau)\partial d_{t-1}$. Next, use the law of motion of government debt to obtain $\partial d_t = \tau\partial d_{t-1} = \tau^t\partial d_0$, which together with the previous expression implies that $\widehat{g}_t = (R - \tau)\tau^{t-1}\partial d_0 = -(R - \tau)\tau^{t-1}\widehat{g}_0$.

¹⁸When there is a monetary expansion at $t = 0$ and τ is very close to one, the money contraction at $t = 1$ is very small. Ideally, for a more “realistic” scenario, one could have a more persistent money expansion, eventually followed by a contraction. In the context of our simple model, since we analyze a one-time money expansion, by using τ very close to one most of the subsequent money contraction occurs several periods after the money expansion.

where $\rho = (a + c - G')\frac{K^*}{M^s/p}$. Notice that prices at $t = 0$ do not change because both output and the money supply used for transactions in the goods market are predetermined. Next, linearizing equation (4) we obtain

$$(14) \quad \hat{q}_t - \beta' \hat{q}_{t+1} = (1 - \beta') \left(\frac{1}{\eta} - \rho \right) \hat{K}_t + (1 - \beta') \rho \hat{K}_{t+1} - (1 - \beta') \hat{g}_{t+1},$$

where $\frac{1}{\eta} = -\frac{G''K^*}{nG'} > 0$.¹⁹ The equation above describes the forward-looking nature of capital prices, i.e., the price of capital at $t = 0$ depends on the full path of capital distributions across types.

Finally, using the three expressions above, as well as the linearized versions of equations (5), (6) and (7), it is easy to show that \hat{K}_t satisfies the following non-homogeneous second order difference equation for $t \geq 2$

$$(15) \quad \theta_0 \hat{K}_t = \theta_1 \hat{K}_{t-1} + \theta_2 \hat{K}_{t-2} + \mu_0 \tau^{t-2} \hat{g}_0,$$

where θ_0 , θ_1 , θ_2 , and μ_0 are constants that depend on steady-state variables (see Appendix B). It can be shown that for β' close to 1, these constants are given by: $\theta_0 = 1 - \rho > 0$, $\theta_1 \approx 2\theta_0$, $\theta_2 \approx -\theta_0$, and $\mu_0 \approx -(1 - \tau)^2$. This last term reflects that a money injection at $t = 0$ generates a negative trend in K_t as a result of the sterilization that takes place after the injection.

The previous equation summarizes the equilibrium dynamics of the model. It can be shown that \hat{K}_t exhibits persistent and dampening cycles, as summarized in the following proposition:

Proposition 2. *For β' sufficiently close to 1 and $\pi = 0$,*

(i) the general solution to (15) is

$$(16) \quad \hat{K}_t = Ar^t \cos(\omega t - \phi) + A_\tau \tau^t \hat{g}_0$$

¹⁹The term $\frac{1}{\eta}$ can be rewritten as $\frac{1}{\eta} = -\frac{G''(\bar{K}-K^*)/n}{G'} \frac{K^*}{\bar{K}-K^*}$, and it can be interpreted as a measure of the elasticity of the marginal product of borrowers' capital, weighted by the ratio of borrowers to lenders' capital in the steady state.

where A and ϕ are constants, $r = \sqrt{-\theta_2/\theta_0}$, $\omega = \cos^{-1}\left(\frac{\theta_1/\theta_0}{2r}\right)$, and $A_\tau = \frac{\mu_0}{\theta_0\tau^2 - \theta_1\tau - \theta_2}$.²⁰

(ii) r is close to, but less than, 1, and ω is close to, but larger than, zero.

Proof: See Appendix B ■

Corollary. \widehat{K}_t exhibits persistent and dampening cycles.

This cyclical dynamics of the model are consistent with the hump-shaped pattern of output response to shocks that has been observed in the data. In order to develop some intuition for the cycles, it is useful to rewrite the dynamics in terms of the two states variables of the model: the borrowers' capital stock, K_{t-1} , and their real money balances, M_{t-1}^r . These real balances are defined as $M_{t-1}^r \equiv M_{t-1}^d/p_t$. Notice that M_{t-1}^r is predetermined at time t because both nominal balances, M_{t-1}^d , and prices are predetermined.²¹ Using (15) and the linearized version of (5) we can write a dynamic system that has the following sign pattern for β' close to 1:²²

$$\begin{bmatrix} \widehat{M}_t^r \\ \widehat{K}_t \end{bmatrix} = \begin{bmatrix} + & - \\ + & + \end{bmatrix} \begin{bmatrix} \widehat{M}_{t-1}^r \\ \widehat{K}_{t-1} \end{bmatrix}.$$

Here the borrowers' capital holdings \widehat{K}_{t-1} act as a "predator", and their real money holdings \widehat{M}_{t-1}^r act as a "prey." A rise in M_{t-1}^r allows borrowers to purchase more capital ($\partial K_t/\partial M_{t-1}^r > 0$). However, a rise in K_{t-1} means that borrowers have a greater debt overhang at date t , which restricts their ability to expand M_t^r ($\partial M_t^r/\partial K_{t-1} < 0$). In other words, capital is a predator because due to the binding collateral constraint, capital is linked strongly to debt. In turn, money acts as the

²⁰Note that $\lim_{\beta' \rightarrow 1} A_\tau = -\frac{1}{(1-\rho)}$.

²¹See the quantity equation (12).

²²It can be shown that the elements of the matrix below are given by: $a_{11} = -\left(1 - \frac{c}{q}\right) + \frac{\theta_1}{\theta_0}$; $a_{12} = \frac{q}{c} \frac{\theta_2}{\theta_0} + \left(1 - \frac{q}{c}\right) \left[\left(1 - \frac{c}{q}\right) - \frac{\theta_1}{\theta_0}\right]$; $a_{21} = \frac{c}{q}$; and $a_{22} = 1 - \frac{c}{q}$.

prey, because higher debt payments restrict borrowers' ability to demand cash in order to buy more capital next period.^{23, 24}

The existence of dampening cycles rather than monotonic dynamics occurs in this model due to the interplay between the CIA and collateral constraints. In particular, the full impact of a shock that increases net worth is delayed in this model because with a binding CIA constraint, collateral can only be accumulated gradually. In other words, the CIA constraint on investment acts as “time to build” constraint for capital. This constraint delays the reaction of the economy to shocks in a way similar to adjustment costs models which are well known to generate cyclical responses. We show this formally in Appendix C. There we show that a model without investment in the CIA constraint but with a time to build restriction exhibits cyclical dynamics similar to the one described in Proposition 2.²⁵

²³The analogy predator-prey is not as clear as in Section III of Kiyotaki and Moore (1997), where they can decouple the system in terms of capital (prey) and debt (predator). We cannot decouple the system in those terms because aggregate capital and debt are strongly linked when the borrowing constraint is binding for all borrowers.

²⁴The representation shown above makes apparent that our monetary model does not exhibit indeterminacy. We have two initial conditions that can be used to solve for A and ϕ in equation (16).

²⁵The fact that it is the time-to-build feature of the CIA what lies behind the cyclical dynamics allows us to draw a parallel between our model, and Section III of Kiyotaki and Moore (1997) where non-collateralizable “trees” are introduced. In their model with trees, cyclical dynamics emerge from the fact that in each period only a fraction of the farmers have an investment opportunity. In a way, investors are exogenously constrained because they have to “wait” for the arrival of the investment opportunity. In our model, investors are constrained through the CIA because they have to “wait” until they collect the cash before they can buy capital.

Furthermore, in Section III of Kiyotaki and Moore (1997) the existence of non-collateralizable trees by itself does not generate cyclical dynamics, but it only dampens the reaction of quantities relative to the price of land. Similarly, in our model, the fact that money is a non-collateralizable asset is not the explanation for cycles, but it only generates a weakening of the leverage effect by reducing the debt to asset ratio.

To fully characterize the equilibrium solution summarized in equation (15), we require two additional conditions on the trajectory of \widehat{K}_t . We first show that $\widehat{K}_0 = 0$ (or $K_0 = K^*$). To see this, consider the borrower's CIA constraint at time zero,

$$q_0^n(k_0 - k^*) + p_0 x_0 \leq m_{-1}^d.$$

Notice that m_{-1}^d was chosen the period before the shock in order to buy consumption goods, i.e., $m_{-1}^d = p_0^e c k^*$, where p_0^e denote the expected price at time zero. In addition, we have already shown (in Appendix A) that around the steady state it is optimal to set $x_t = c k_{t-1}$. Therefore, the equation above can be written as

$$q_0^n(k_0 - k^*) \leq (p_0^e - p_0) c k^*.$$

Moreover, according to the quantity equation (12), p_0 depends only on M_{-1}^s and K^* so that $p_0^e = p_0$. In particular, the monetary injection does not affect time-zero prices because the injection occurs in the bonds market, not in the goods market. As a result, k_0 must satisfy $q_0^n k_0 \leq q_0^n k^*$. Finally, we have already shown (in Appendix A) that it is optimal for borrowers to invest all resources around the steady state. Thus, $k_0 = k^*$ and $K_0 = K^*$.

Since $\widehat{K}_0 = 0$, then the equilibrium path can be completely characterized in terms of \widehat{K}_1 . Given \widehat{K}_1 and $\widehat{K}_0 = 0$, one can use equation (16) evaluated at $t = 0$ and $t = 1$ to solve for A and ϕ . The solution is

$$A = \frac{\widehat{K}_1 - A_\tau \tau \widehat{g}_0}{\cos(\omega - \phi)} \text{ and } \cos(\phi) = -\frac{A_\tau \widehat{g}_0}{A}.$$

We now solve for \widehat{K}_1 following a monetary shock at $t = 0$. For this purpose, combine (5), (6), and (7) to obtain

$$(17) \quad q_1(K_1 - K_0) + cK_0 = \frac{a + c}{1 + \pi_0} K_{-1} + \frac{1}{1 + \pi_0} \frac{B_0}{p_0} - \frac{R_0}{(1 + \pi_0)(1 + \pi_{-1})} \frac{B_{-1}}{p_{-1}},$$

where $\frac{B_{-1}}{p_{-1}}$ is the aggregate steady-state level of debt in real terms, and K_{-1} corresponds to the borrowers' steady-state capital level.

We consider two relevant cases at this point, non-indexed (or nominal) versus indexed debt contracts. The difference between these two is that under indexed contracts, borrowers must compensate lenders for any unexpected inflation. Thus, debt repayments at time zero are immune to inflation π_0 , i.e., $R_0 B_{-1} = \frac{1+\pi_0}{\beta'} B_{-1}$. In contrast, under nominal financial contracts, unexpected inflation/deflation produces a redistribution of wealth between debtors and creditors, as emphasized by Fisher (1933).

The focus of this paper is to study the transmission of money shocks in the presence of nominal financial contracts, the Fisher effect, and indexation. If contracts are indexed, borrowers' nominal liabilities are proportional to the price level. Here, we do not consider fully state-contingent contracts under which these liabilities can be *any* function of the price level, and the collateral constraint will be set state by state.²⁶

3.1. Nominal contracts

When debt contracts are nominal or non-indexed, R_0 in (17) is simply the steady-state nominal interest rate $R = \frac{1}{\beta'}$. Linearizing (17) in this case yields

$$(18) \quad \widehat{K}_1 = \frac{1}{1 - \beta' \rho} \left[\beta' \widehat{q}_1 + (2 - r^h - \beta' \tau) \varepsilon \right].$$

To solve for \widehat{K}_1 we first need to solve for \widehat{q}_1 , which in turn depends on the full sequence $\{\widehat{K}_t\}$. Solving equation (14) forward we can obtain a solution for \widehat{q}_1 , as shown in Appendix D. The expression that relates \widehat{q}_1 with \widehat{K}_1 is algebraically complicated.

We can use equation (18) to gain some intuition on the real effects of a monetary expansion $\varepsilon > 0$ under non-indexed contracts. Suppose initially that the real price of

²⁶We thank an anonymous referee for pointing this out to us. We analyze the effects of nominal vs. indexed contracts without asking which of the two types of contracts would emerge as optimal. For optimal state-contingent contracts in simple non-monetary economies see Krishnamurthy (2000) and Lorenzoni (2002).

capital remains unchanged after the monetary shock so that $\widehat{q}_1 = 0$. In this case, the real effects of the shock depend on the size of τ . Specifically, if $\tau < \bar{\tau} \equiv \frac{1}{\beta'} (2 - r^h)$, then K_1 , and also Y_2 , move in the same direction as the monetary shock. This condition is satisfied when β' is close to 1 because in that case $\bar{\tau}$ is close to 2. Finally, this change in the distribution of capital toward the more productive agents induces an increase in the price of capital, $\widehat{q}_1 > 0$, which reinforces the initial effect of the shock. Therefore, a monetary expansion under non-indexed debt contracts induces a redistribution of capital towards borrowers, and increases output.

3.2. Indexed contracts

As indicated above, if debt contracts are indexed, then $R_0 = \frac{1+\pi_0}{\beta'}$. In this case, linearization of equation (17) yields

$$(19) \quad \widehat{K}_1 = \frac{1}{1 - \beta'\rho} \left[\beta' \widehat{q}_1 + (1 - r^h - \beta'\tau) \varepsilon \right].$$

The equation above is very similar to (18), except that now we have a smaller coefficient on ε . This smaller coefficient has two important implications. First, the real effects of the money shock will be smaller than in the case of non-indexed debt contracts, i.e., the redistribution of capital as well as the output amplification are smaller. Second, in contrast with the case of non-indexed debt contracts, a monetary expansion may now produce an output *downturn* rather than a boom. To see why, assume for a moment that $\widehat{q}_1 = 0$. In that case, the real effects of the shock again depend on the size of τ . If $\tau < \bar{\tau} \equiv \frac{1}{\beta'} (1 - r^h)$, then the monetary expansion will increase output. But for β' close to 1, $\bar{\tau}$ is close to 1. Thus, it may be possible to find $\tau > \bar{\tau}$ such that the monetary expansion generates a downturn.

To better highlight the mechanisms behind this result, it is useful to rewrite equation (17) as

$$(20) \quad q_1(K_1 - K_0) + cK_0 = \frac{a+c}{1+\pi_0} K_{-1} + \frac{\beta' q_1 K_0}{1+\pi_1} - \frac{R_0}{(1+\pi_0)(1+\pi_{-1})} \frac{B_{-1}}{p_{-1}},$$

where the left-hand side represents consumption and investment in $t = 1$, and the right-hand side are the real balances brought from period $t = 0$. In particular, the first term on the right-hand side are output sales; the second term is new debt contracted in $t = 0$; and the third term is the repayment for debt contracted in $t = -1$. Notice that consumption in $t = 1$ is fixed because K_0 remains at the steady-state level. Thus, the only way borrowers increase their investment in capital K_1 , is if the right-hand side of the equation is larger than its steady-state value.

First, notice that under indexed debt contracts, since $R_0 = \frac{1+\pi_0}{\beta'}$, then the last term on the right-hand side does not change with the money expansion, i.e., it remains at its steady-state value. This implies that lenders do not transfer any wealth to borrowers in the period of the shock via interest rate repayments.

Second, due to the monetary expansion $\hat{g}_0 > 0$, the level of prices in $t = 1$ increases, so that π_0 increases ($\hat{v}_0 > 0$). This hurts borrowers because the first term on the right-hand side $\frac{a+c}{1+\pi_0}K_{-1}$ decreases. Thus, the only way borrowers could increase K_1 is if the second term on the right-hand side $\frac{\beta'q_1K_0}{1+\pi_1}$ increases by more than the decrease in $\frac{a+c}{1+\pi_0}K_{-1}$. In general, π_1 decreases ($\hat{v}_1 < 0$) due to the policy reversal, i.e., the money contraction that follows the expansion. However, for τ sufficiently close to 1, case in which the money contraction in $t = 1$ is very small, the drop in π_1 is so small, that $\frac{\beta'q_1K_0}{1+\pi_1}$ increases very little. In this case, borrowers decrease K_1 and the money expansion causes a downturn. In summary, the only way a monetary expansion can generate an increase in output when debt contracts are indexed is when this expansion is quickly reverted by a monetary contraction (i.e., small τ).

Finally, notice that when debt is non-indexed, the third term on the right-hand side of (20) decreases with respect to the steady state because π_0 increases due to the money expansion. What this means is that there is a transfer of wealth from creditors to debtors, which ultimately allows borrowers to increase K_1 and generate an output

expansion.

3.3. Default and asymmetric business cycles

Up to now we have assumed that the shock arrives in the morning when it is too late to escape. Borrowers thus never have incentives to repudiate their debt contracts in the face of a shock because lenders could seize their capital, production, and money. Consider now what would happen if the shock is known when borrowers still have a chance to escape. For example, the government announces, after all markets are closed for the day, that an unexpected open-market operation will take place the next day. In this case, if the value of the collateral is expected to fall below the value of the debt, then borrowers could find it optimal to escape overnight. Alternatively, borrowers could be able to renegotiate their debt down to the market value of the collateral. It turns out that if such renegotiation is possible, then our model can generate asymmetric business cycles. In particular, if contracts are non-indexed and there is a monetary *contraction*, borrowers have incentives to renegotiate their debt and the output downturn is smoother.

The intuition for this result is as follows. If contracts are non-indexed, the interest rate borrowers pay is the steady-state rate. If there is a money contraction, this steady-state rate will be higher than the equilibrium rate, due to deflation. Thus, borrower's net worth is reduced, as well as their capital holdings. This in turn triggers a decrease in the price of capital. In this case, borrowers have incentives to repudiate their debts and pay back just the market value of the collateral. The fact that borrowers end up paying back less makes the output downturn less severe. Notice that in contrast, if there is a monetary expansion, borrowers will have no incentive to repudiate and renegotiate their debts. This is so because in this case the price of capital increases, while the nominal interest rate remains at its steady state value.²⁷

²⁷If debt contracts are indexed, renegotiation occurs depending on how interest payments change

Analytically, when debt is renegotiated, debt repayments, $R_0 \frac{B_{-1}}{p-1}$, are equal to the market value of the collateral, $q_0 K_{-1}$. Linearizing equation (17) in this case yields

$$\widehat{K}_1 = \frac{1}{1 - \beta' \rho} \left[\beta' \widehat{q}_1 - \widehat{q}_0 + (2 - r^h - \beta' \tau) \varepsilon \right]$$

which holds for both the indexed and non-indexed debt cases. It turns out that when debt is renegotiated, the solution for \widehat{K}_1 can be simply written as²⁸

$$(21) \quad \widehat{K}_1 = \frac{1}{\theta_0} \left[(1 - r^h) - (1 - 2\beta') (R - \tau) \right] \widehat{g}_0.$$

Since the expression above is algebraically simple, we can use it to analyze whether following the monetary contraction in period $t = 0$, it is the case that $\widehat{K}_1 < 0$ and so $\widehat{Y}_2 < 0$. Further, if $\widehat{K}_2 < \widehat{K}_1$, since the model exhibits persistent dampening cycles, we should observe a downturn in the economic activity as borrowers' capital level decreases. Proposition 3 summarizes the conditions under which these results hold. Let $\alpha \equiv \frac{cK^*}{Y^*} < 1$ be the fraction of steady-state output consumed by the borrowers.

Proposition 3. *For β' sufficiently close to 1 and $\pi = 0$,*

(i) *following a decrease in the money growth rate $\varepsilon < 0$ at $t = 0$, borrowers decrease their capital holdings in period $t = 1$, i.e., $\widehat{K}_1 < 0$. Further, the lower τ , the larger $|\widehat{K}_1|$ is.*

(ii) *if τ is sufficiently close to 1 then $\widehat{K}_2 < \widehat{K}_1$, while if $\tau \rightarrow 0$ then a sufficient condition for $\widehat{K}_2 < \widehat{K}_1$ is that $\alpha < \frac{1}{3}$.*

Proof: (i) When $\beta' \rightarrow 1$ it is the case that $\rho \rightarrow \alpha$ and that $r^h \rightarrow 0$. Then,

$\theta_0 \rightarrow (1 - \alpha)$. Thus, when $\beta' \rightarrow 1$ from equation (21) we have that: $\widehat{K}_1 \rightarrow \frac{2 - \tau}{1 - \alpha} \widehat{g}_0$, compared to the change in the price of the collateral. For instance, suppose τ is small enough so that a monetary expansion triggers an increase in output. Then, if the nominal interest rate increases by more than the increase in the price of capital, borrowers will have incentives to renegotiate.

²⁸Under renegotiation, the solution for \widehat{K}_1 is the same as that implied by the non-homogeneous second order differential equation for $t = 1$ and $\widehat{K}_0 = 0$.

and since $\tau < 1$ and $\widehat{g}_0 < 0$ it follows that $\widehat{K}_1 < 0$. Notice that the more slowly government debt returns to the steady state, i.e., the larger τ , the lower the multiplier of monetary policy in the first period.

(ii) From equation (15) we have: $\widehat{K}_2 = \frac{\theta_1}{\theta_0} \widehat{K}_1 + \frac{\mu_0}{\theta_0} \widehat{g}_0$, and since when $\beta' \rightarrow 1$ we have that $R \rightarrow 1$ and so $\mu_0 \rightarrow -(1 - \tau)^2$, then: $\widehat{K}_2 \rightarrow \frac{1}{1-\alpha} \left[\frac{2-\tau}{1-\alpha} - (1-\tau)^2 \right] \widehat{g}_0$. If $\tau \rightarrow 1$, then $\widehat{K}_2 \rightarrow \frac{1}{1-\alpha} \widehat{K}_1$ and so $\widehat{K}_2 < \widehat{K}_1$. On the other hand, if $\tau \rightarrow 0$, then $\widehat{K}_2 \rightarrow \left[\frac{1}{1-\alpha} - \frac{1}{2} \right] \widehat{K}_1$, so that $\widehat{K}_2 < \widehat{K}_1$ if $\alpha < \frac{1}{3}$ ■

Part (ii) in Proposition 3 indicates the role of τ in strengthening the real effects of a monetary contraction. In fact, when the sterilization policy is smooth, i.e., when τ is large, borrowers further decrease their capital stock in $t = 2$. This implies that they would be able to borrow less against their collateral, and their capital holdings will decrease for a number of periods after the shock. This occurs because when the sterilization is smooth, then the government expands the money supply in small amounts during several periods, and so the nominal interest rate remains above the steady state, i.e., $\widehat{R}_t > 0$ for a longer time. In contrast, when the monetary contraction is reversed quickly, i.e., when $\tau \rightarrow 0$, this dynamic pattern for capital may not necessarily hold, unless further conditions are imposed.

4. Simulations

To illustrate the magnitude and persistence of monetary shocks in this economy, we assign values to the parameters of the economy and simulate the effects of a 1% change in the growth rate of money at $t = 0$. The only purpose here is to illustrate the dynamics generated by our model, not to calibrate our stylized model. As such, the quantitative results presented here are not to be taken literally. We choose the parameters of the model to satisfy the assumptions imposed. We set $\beta' = 0.995$ to

simulate a time period equal to a month. Note that β' is sufficiently close to 1, to accord with the proofs presented above.

We normalize to unity the total stock of capital. The production technology for lenders is: $G(K) = B(\bar{K} - K)^\gamma$, where B is also normalized to unity, and $\gamma = 0.2$. We set $n = 3$, which implies that in this economy only 25% of the agents are constrained. Finally, we choose $c = 1/2$ and a such that lenders hold 50% of the total capital in steady state.

Figure 2 displays the effects of an unanticipated increase of 1% in the growth rate of money when $\pi = 0$, $\tau = 0.9$, and debt contracts are non-indexed. The figure shows percentage deviations from steady-state values. Since τ is large, the subsequent money contraction is smooth and government bonds go back gradually to their steady-state $H^s = 0$. As is shown in the graph, this policy generates ample and persistent dampening cycles. The cycle starts with an increase in borrowers' capital holdings, as well as an increase in output. The peak of the cycle is reached about 25 months after the shock, when output is around 80% above the steady state. Of course, this quantitative result is unrealistically large and due to non-standard assumptions of the model such as linear utility and linear production function for borrowers. What lies behind such large amplification is the redistribution of wealth from lenders to borrowers due to the nominal nature of the debt contract. In the figure, since the collateral constraint binds, real borrowers' debt mimics the behavior of capital.

These results emerge from the combination of two mechanisms that affect both sides of the collateral constraint: one is the asset-price effect, and the other is the interest-rate effect. First, in the period of the shock there is an increase in real price of capital of about 60% that increases the value of the collateral for a number of periods. This increase in the asset price comes from the fact that to clear the capital market, the users cost for *lenders* has to increase. Notice that the real price of capital

is above the steady state for 25 months, which is exactly the time at which capital and bonds reach their peaks. Second, the nominal interest rate is at its steady-state value in the period of the shock, but it then decreases below the steady state.

Finally, borrowers' money demand and net worth mimic the pattern of capital prices: they are above the steady state while borrowers are investing, and below the steady state while they disinvest. Thus, in this model disinvestment temporarily reduces liquidity needs. Notice that borrowers' net worth increases in the period of the shock because the value of capital increases, and also because unexpected inflation under non-indexed contracts reduces real debt repayments.

Figure 3 displays the effects of an unanticipated increase of 1% in the growth rate of money when $\pi = 0$, $\tau = 0.9$, and debt contracts are indexed. The most striking feature of Figure 3 is that the real effects of the money shock under indexed contracts are small when compared to those obtained in Figure 2. This highlights the difference between nominal and indexed contracts. When debt is indexed, there is no redistribution of wealth from less-productive lenders to more-productive borrowers. Moreover, the asset-price effect under indexed contracts is very small. Net worth decreases dramatically in the period of the shock because unexpected inflation under fully indexed contracts increases debt repayments.²⁹

Finally, Figure 4 illustrates the case in which debt renegotiation takes place. It displays the effects of a decrease of 1% in the growth rate of money when $\pi = 0$, $\tau = 0.9$, and debt is non-indexed. In this case, the model still exhibits persistence, and the amplitude of the effects is much smaller than those observed in Figure 2. Notice that output reaches a trough of about -1.5% . Renegotiation avoids a deep economic downturn because it partially protects borrowers from deflation. In fact, in

²⁹Although it is not clear from Figure 3, borrowers' net worth also cycles around the steady state. In this case, since there is indexation of contracts, deviations from the steady state are small compared with the substantial decrease in net worth observed in the period of the shock.

the period of the shock net worth remains at its steady-state value because borrowers are able to renegotiate debt payments down to the market value of collateral.

5. Concluding comments

This paper analyzes the propagation of monetary shocks by combining collateral and cash-in-advance constraints in a world in which changes in the money supply occur via open-market operations. We find that an unanticipated monetary injection generates persistent movements in aggregate output, the amplitude of which depends upon whether debt contracts are nominal or indexed. In general, output fluctuations are larger if only nominal contracts exist. Due to the interaction between the cash-in-advance and collateral constraints, monetary shocks trigger a highly persistent dampening *cycle* rather than a smoothly declining deviation.

The is sufficiently simple to provide insights on how collateral constraints work in a monetary economy. Since the model is highly stylized, future work can involve the following extensions. First, both the utility and production functions may be modified to a more standard concave specification. This would be particularly important if the mechanisms described here were to be carefully quantified and compared with data. Second, in order to avoid excess response in nominal prices, other frictions besides collateral constraints would need to be introduced. Finally, the model has not been carefully calibrated to assess its ability to match the data. A careful calibration exercise is beyond the scope of this paper and is left for future work.

In summary, what we learn from the simple, stylized model analyzed here is that collateral constraints, in combination with cash-in-advance constraints, constitute a potential mechanism that can transform small monetary shocks into persistent output fluctuations.

Appendix

A. Proof of optimal solution for borrowers

We need to prove the claim that borrowers' optimal plan is to consume only the nontradable fraction of output, i.e., $x_t = cK_{t-1}$, to borrow up to the limit and to invest all remaining resources. To do that we compare the utility achieved under the different alternative plans. The first one is to follow the proposed *investment path*. Alternatively, borrowers can consume or save. For these last two alternatives, we only consider single deviations from the investment path at date $t = 0$.³⁰

Consider the borrower's marginal utility of investing p_0 dollars given that all aggregate variables remain unchanged at their steady state levels. For simplicity let $\pi = 0$. In steady state, we have $R = 1/\beta'$ and $q = a/(1 - \beta')$. Therefore, for given prices and aggregate variables at their steady state levels, equations (5), (6) and (7) can be rewritten as

$$(A1) \quad qk_t + (c - q)k_{t-1} = \frac{m_{t-1}^d}{p_{t-1}},$$

$$(A2) \quad \frac{b_t}{p_t} = q\beta'k_t,$$

$$(A3) \quad \frac{m_t^d}{p_t} = (a + c)k_{t-1} + \frac{b_t}{p_t} - R\frac{b_{t-1}}{p_{t-1}}.$$

Replacing the borrowing constraint into the budget constraint,

$$(A4) \quad \frac{m_t^d}{p_t} = (a + c - q)k_{t-1} + q\beta'k_t.$$

³⁰Following the logic of Kiyotaki and Moore (1997), "we appeal to the principle of unimprovability", which states that to prove that our proposed strategy of investing all the extra p_0 dollars is optimal, we need to consider only single deviations from this plan at date $t = 0$.

Substituting (A4) into (A1) and solving for k_t

$$(A5) \quad k_t = \left[\beta' + 1 - \frac{c}{q} \right] k_{t-1} + \left[\frac{a+c}{q} - 1 \right] k_{t-2}.$$

From the steady state value of q we have that $\left(\beta' + 1 - \frac{c}{q} \right) = 2 - (1 - \beta') - \frac{c}{q} = 2 - \frac{a+c}{q} = 2 - r^h$. Let $r^h \equiv \frac{a+c}{q}$. We can rewrite (A5) as

$$(A6) \quad k_t = \left(2 - r^h \right) k_{t-1} + (r^h - 1)k_{t-2}.$$

It is easy to check that the roots of the associated characteristic polynomial are 1 and $1 - r^h$. Therefore, k_t can be expressed as

$$(A7) \quad k_t = A_1 + A_2(1 - r^h)^t.$$

where constants A_1 and A_1 need to be determined. Under the proposed guess, the optimal strategy for borrowers is to use the extra p_0 dollars to invest in capital. With this amount, the borrower can buy $k_0 = 1/q$ units of capital at $t = 0$. This allows him to borrow $q\beta'k_0 = \beta'$ additional units of output.³¹ At $t = 1$, consumption increases by ck_0 units so that from the additional resources, $\beta' - c/q$ can be used to buy capital. Therefore, investment is given by: $k_1 - k_0 = \frac{\beta' - c/q}{q}$, so that $k_1 = k_0(\beta' - c/q + 1)$. With these two initial values (k_0, k_1) , constants A_1 and A_2 can be determined as follows

$$A_2 = \frac{k_0 - k_1}{r^h} \text{ and } A_1 = \frac{1}{r^h} \left[k_1 - (1 - r^h)k_0 \right].$$

Utility under the investment path is given by

$$\begin{aligned} U^{inv} &= \beta c \sum_{t=0}^{\infty} \beta^t k_t = \beta c \sum_{t=0}^{\infty} \beta^t \left[A_1 + A_2(1 - r^h)^t \right] = cA_1 \frac{\beta}{1 - \beta} + cA_2 \frac{\beta}{1 - \beta(1 - r^h)} \\ &= \frac{c}{q} \frac{\beta}{1 - \beta} \frac{1}{1 - \beta(1 - r^h)}. \end{aligned}$$

³¹Note that p_0 dollars are equivalent to one unit of output at $t = 0$ prices. Also, by borrowing extra $b_0 = \beta'$, the agent can demand extra β' real money balances in the third subperiod of $t = 0$, in order to buy additional capital in the first subperiod of $t = 1$.

To show that higher utility is attained in the investment path than in the *consumption path*, we need to find conditions under which

$$\frac{c}{q} \frac{\beta}{1-\beta} \frac{1}{1-\beta(1-r^h)} > 1.$$

We can transform the expression above to obtain

$$\frac{c}{a} > \frac{(1-\beta)}{\beta} \frac{(1-\beta)}{(1-\beta')} + (1-\beta) \frac{a+c}{a}.$$

Since $\frac{1-\beta}{\beta} < \frac{1-\beta}{\beta^2}$ then a sufficient condition for the utility from the investment path being higher is

$$\frac{c}{a} > \frac{(1-\beta)}{\beta^2} \left[\frac{(1-\beta)}{(1-\beta')} + 1 \right],$$

which corresponds to Assumption 2 in the text.

To complete the proof we need to show that higher utility is attained in the investment path than in the *saving path*. Borrowers can save the p_0 dollars and use the return R to commence a strategy of maximum levered investment from date $t = 1$ onwards. Then, all we need to show is that the returns from saving p_0 dollars in period $t = 0$ are lower than the return from investing at $t = 0$. Since from Assumption 1, $\beta' > \beta$, using Assumption 2 is easy to show that $\beta' > \frac{a}{a+c}$. Thus,

$$1 + r^h = 1 + \frac{a+c}{q} = 1 + \frac{(a+c)(1-\beta')}{a} > 1 + \frac{1-\beta'}{\beta'} = \frac{1}{\beta'} = R.$$

Therefore, $1 + r^h > R$, which guarantees that the investment path yields more utility than the alternative savings path. This completes the proof that the proposed solution is an equilibrium. We have presented an analytical proof for $\pi = 0$. For $\pi \neq 0$ it is not possible to provide an analytical proof. However, for all the numerical simulations in the text, we have verified in the computer that the decision rules for the borrower are optimal.

B. Proof of Proposition 2

Let $\pi = 0$ so that in steady state $u = a$. Equation (15) in the text reads

$$(B1) \quad \theta_0 \widehat{K}_t = \theta_1 \widehat{K}_{t-1} + \theta_2 \widehat{K}_{t-2} + \mu_0 \tau^{t-2} \widehat{g}_0,$$

where

$$\begin{aligned} \theta_0 &= 1 + (1 - 2\beta')\rho, \\ \theta_1 &= (1 - r^h)(1 - \rho) + 1 + (1 - 2\beta')\rho - (1 - \beta')\frac{1}{\eta}, \\ \theta_2 &= -(1 - r^h)(1 - \rho), \\ \mu_0 &= -(R - \tau) \left[(1 - 2\beta')\tau + (1 - r^h) \right]. \end{aligned}$$

Since the particular solution for the equation above is

$$\widehat{K}_p = \frac{\mu_0 \widehat{g}_0 \tau^t}{\theta_0 \tau^2 - \theta_1 \tau - \theta_2},$$

then the general solution is given by

$$(B2) \quad \widehat{K}_t = A_1 \lambda_1^t + A_2 \lambda_2^t + A_\tau \tau^t \widehat{g}_0,$$

where $A_\tau = \frac{\mu_0}{\theta_0 \tau^2 - \theta_1 \tau - \theta_2}$ is a constant and the eigenvalues λ_1 and λ_2 satisfy: $\lambda_1 \lambda_2 = \frac{-\theta_2}{\theta_0}$ and $\lambda_1 + \lambda_2 = \frac{\theta_1}{\theta_0}$. Finally, the solutions for constants A_1 and A_2 can be obtained from: $\widehat{K}_1 = A_1 \lambda_1 + A_2 \lambda_2 + A_\tau \tau \widehat{g}_0$ and $\widehat{K}_0 = A_1 + A_2 + A_\tau \widehat{g}_0$.

B.1. Cycles

The dynamic properties of equation (B1) depend on the eigenvalues associated to the homogeneous difference equation $\theta_0 \widehat{K}_t = \theta_1 \widehat{K}_{t-1} + \theta_2 \widehat{K}_{t-2}$ which are given by

$$\lambda_1, \lambda_2 = \frac{\theta_1 \pm \sqrt{\theta_1^2 + 4\theta_0 \theta_2}}{2\theta_0}.$$

The necessary and sufficient condition for cycles is $\theta_1^2 + 4\theta_0\theta_2 < 0$. Note that θ_1 can be rewritten as

$$(B3) \quad \theta_1 = \theta_0 - \theta_2 - (1 - \beta') \frac{1}{\eta}.$$

Adding and subtracting proper terms, θ_2 can be rewritten as

$$(B4) \quad \theta_2 = \xi(\beta') - \theta_0,$$

where

$$\begin{aligned} \xi(\beta') &\equiv 2\rho(1 - \beta') + r^h(1 - \rho) \\ &= (1 - \beta')(2\rho + \frac{a+c}{a}(1 - \rho)) > 0. \end{aligned}$$

From (B3) and (B4), θ_1 can be written as

$$(B5) \quad \theta_1 = 2\theta_0 - \zeta(\beta'),$$

where

$$\begin{aligned} \zeta(\beta') &\equiv (1 - \beta')(2\rho + \frac{a+c}{a}(1 - \rho) + \frac{1}{\eta}) \\ &< (1 - \beta') \left[\max \left\{ 2, \frac{a+c}{a} \right\} + \frac{1}{\eta} \right]. \end{aligned}$$

Finally, from (B3) and (B4), $\lim_{\beta' \rightarrow 1} \theta_2 = \theta_0$ and $\lim_{\beta' \rightarrow 1} \theta_1 = 2\theta_0$.

B.2. Proof of Proposition 3

To show that for β' sufficiently large the model exhibits cycles, it needs to be proven that $\theta_1^2 + 4\theta_0\theta_2 < 0$. Use (B4) and (B5) to get

$$\begin{aligned} \theta_1^2 + 4\theta_0\theta_2 &= (2\theta_0 - \zeta(\beta'))^2 - 4[\theta_0 - \xi(\beta')] \theta_0 \\ &< -4\theta_0(1 - \beta') \frac{1}{\eta} + (1 - \beta')^2 \left[\max \left\{ 2, \frac{a+c}{a} \right\} + \frac{1}{\eta} \right]^2. \end{aligned}$$

Note that $\lim_{\beta' \rightarrow 1} \frac{1}{\eta} = -\frac{G''((\bar{K}-K^1)/n)K^1}{nG'((\bar{K}-K^1)/n)} > 0$ where K^1 is the solution of (13) for β' equal to 1 and $\pi = 0$. Therefore, the second term in the last expression approaches to zero faster than the first term as $\beta' \rightarrow 1$. Note that $\theta_0 \frac{1}{\eta}$ remains bounded above since θ_0 approaches $1 - \alpha^N(K^1) > 0$ and the fact that $\frac{1}{\eta}$ approaches a constant greater than zero. Thus, for β' large enough the first term dominates and the expression is negative ■

It is also useful to state solution (B2) in its polar representation (See Allen, 1959, page 189)

$$\widehat{K}_t = Ar^t \cos(\omega t + \phi) + A_\tau \tau^t,$$

where A and ϕ are constants that can be determined from the initial conditions, and

$$r = \sqrt{-\theta_2/\theta_0},$$

$$\omega = \cos^{-1}\left(\frac{\theta_1/\theta_0}{2r}\right).$$

Stability is guaranteed if the modulo r is less than 1, a result that follows from (B5) for large β' . In addition, r is close to 1 when β' is close to 1. Thus, the difference equation displays persistent dampening cycles.

C. CIA in investment as time-to-build capital

We show that a model in which: (1) investment is not in the CIA constraint; and (2) there is a time-to-build constraint for capital accumulation, generates a cyclical response like the one described in Proposition 2. We proceed in two steps: first, we show that changes in capital prices are fundamental in obtaining cyclical dynamics but changes in the interest rates are not. This result allows us to abstract from

changes in the nominal interest rate.³² Second, we solve the model proposed above.

Step 1 Equation (A6) shows that when prices (the nominal interest rate and capital prices) are constant, cyclical dynamics cannot emerge in our model. Next, if capital prices are kept fixed and interest rates are allowed to adjust, (A6) is transformed to

$$\widehat{K}_t = (1 + \phi) \widehat{K}_{t-1} - \phi \widehat{K}_{t-2},$$

where $\phi \equiv \frac{(1-r^h)(1-\rho)}{1-\beta'\rho}$. Again, cyclical dynamics cannot emerge in this case because the associated roots are 1 and ϕ .

In contrast, if the interest rate is kept fixed and capital prices are allowed to adjust, it can be shown that the dynamics of the model are described by

$$\widehat{K}_t = (2 - r^h - \epsilon) \widehat{K}_{t-1} + (r^h - 1) \widehat{K}_{t-2},$$

where $\epsilon \equiv \frac{1-\beta'}{\eta}$, and the associated roots are given by

$$\left(1 - \frac{r^h + \epsilon}{2}\right) \pm \frac{1}{2} \sqrt{(r^h + \epsilon)^2 - 4\epsilon}.$$

For β' sufficiently large, ϵ and r^h are close to zero, and the term in the square root is negative, giving rise to cyclical dynamics. Since in addition the real part of the roots is less than one, the model exhibits dampening oscillations. Furthermore, it can be shown that for β' sufficiently large, the general equilibrium equation for capital behaves in the same way as the partial equilibrium equation with fixed interest rates. In conclusion, since changes in the interest rate play no role in generating the cyclical behavior of the model, we abstract from them in what follows.

Step 2 Consider the following model in which: (1) investment does not enter the CIA constraint; and (2) in order to introduce some time-to-build, at time t , capital

³²If the interest rate is allowed to adjust similar results hold, but we can only verify them numerically.

for period $t + 1$ is decided. The CIA and budget constraints for lenders in this model are given by

$$p_t x'_t \leq m_{t-1}^d,$$

$$m_t^d + R_t b'_{t-1} + h'_t + q_t^n (k'_{t+1} - k'_t) \leq p_t G(k'_{t-1}) + b'_t + R_t h'_{t-1},$$

which imply the following equilibrium nominal interest rate

$$R_t = \frac{p_{t+1}}{\beta' p_t} \equiv \frac{1 + \pi_t}{\beta'},$$

and a users costs for lenders given by

$$u'_t \equiv \frac{q_t}{1 + \pi_t} - \beta' \frac{q_{t+1}}{1 + \pi_{t+1}} = \beta'^2 \frac{1}{1 + \pi_{t+2}} G'(k'_{t+1}).$$

Similarly, the CIA, budget and collateral constraints for borrowers are respectively given by

$$p_t x_t \leq m_{t-1}^d,$$

$$m_t^d + R_t b_{t-1} + q_t^n (k_{t+1} - k_t) + h_t \leq (a + c)p_t k_{t-1} + b_t + R_t h_{t-1},$$

$$R_{t+1} b_t \leq q_{t+1}^n k_t.$$

These equations, along with optimality conditions imply that

$$k_{t+1} = (1 + \pi_t) \left[\beta' \frac{q_{t+1}}{q_t} \frac{1}{1 + \pi_{t+1}} - \frac{c}{q_t} + \frac{1}{1 + \pi_t} \right] k_t + \left[\frac{a + c}{q_t} - 1 \right] k_{t-1}.$$

It can be shown that for fixed interest rates, the linearized version of the equation above is given by³³

$$\widehat{K}_t = \frac{2 - r^h}{1 + \epsilon} \widehat{K}_{t-1} + \frac{r^h - 1}{1 + \epsilon} \widehat{K}_{t-2},$$

which associated roots are

$$\frac{2 - r^h}{2(1 + \epsilon)} \pm \frac{1}{2(1 + \epsilon)} \sqrt{(r^h)^2 - 4\epsilon(1 - r^h)},$$

³³If interest rates are allowed to adjust, the linearized dynamic equation would be an AR(3) rather than an AR(2). Since analytical results are more complicated with an AR(3), we have verified numerically that cyclical dynamics also obtain in this case.

and since as $\beta \rightarrow 1$, r^h goes to zero, the term in the square root is negative, and we have cyclical dynamics as desired.

D. Forward looking solution for asset prices

This appendix gives the solution for \hat{q}_0 and $\pi = 0$. From equation (14) in the text

$$\hat{q}_t - \beta' \hat{q}_{t+1} = (1 - \beta') \left(\frac{1}{\eta} - \rho \right) \hat{K}_t + (1 - \beta') \rho \hat{K}_{t+1} - (1 - \beta') \hat{g}_{t+1},$$

iterate forward and use the transversality condition $\hat{q}_\infty = 0$ to rule out bubbles in the price of capital to obtain

$$\begin{aligned} \frac{\hat{q}_0}{(1 - \beta')} &= \sum_{j=0}^{\infty} \beta'^j \left[\left(\frac{1}{\eta} - \rho \right) \hat{K}_j + \rho \hat{K}_{j+1} - \hat{g}_{j+1} \right] \\ &= \left(\frac{1}{\eta} - \rho \right) \hat{K}_0 + \sum_{j=0}^{\infty} \beta'^j \left(\beta' \left(\frac{1}{\eta} - \rho \right) + \rho \right) \hat{K}_{j+1} - \sum_{j=0}^{\infty} \beta'^j \hat{g}_{j+1}. \end{aligned}$$

Using the solution for the non-homogeneous second order difference equation (B1)

we have

$$\begin{aligned} \frac{\hat{q}_0}{(1 - \beta')} &= \left(\frac{1}{\eta} - \rho \right) \hat{K}_0 + \left(\beta' \left(\frac{1}{\eta} - \rho \right) + \rho \right) \sum_{j=0}^{\infty} \beta'^j \left(A_1 \lambda_1^{j+1} + A_2 \lambda_2^{j+1} + A_\tau \tau^{j+1} \hat{g}_0 \right) \\ &\quad + (R - \tau) \hat{g}_0 \sum_{j=0}^{\infty} \beta'^j \tau^j, \end{aligned}$$

which after some algebra yields

$$\begin{aligned} \frac{\hat{q}_0}{(1 - \beta')} &= \left(\frac{1}{\eta} - \rho \right) \hat{K}_0 + \left(\beta' \left(\frac{1}{\eta} - \rho \right) + \rho \right) \frac{(\lambda_1 A_1 + \lambda_2 A_2) - \beta' \lambda_1 \lambda_2 (A_1 + A_2)}{1 - \beta' (\lambda_1 + \lambda_2) + \beta'^2 \lambda_2 \lambda_2} \\ &\quad + \left(\beta' \left(\frac{1}{\eta} - \rho \right) + \rho \right) \frac{\tau A_\tau \hat{g}_0}{1 - \beta' \tau} + \frac{(R - \tau) \hat{g}_0}{1 - \beta' \tau}. \end{aligned}$$

Finally using $\hat{K}_0 = 0$ and the properties of λ_1 , λ_2 and \hat{K}_1 from Appendix B we get, after some algebra

$$\begin{aligned} \frac{\hat{q}_0}{(1 - \beta')} &= \frac{\left(\beta' \left(\frac{1}{\eta} - \rho \right) + \rho \right) \theta_0}{\theta_0 - \beta' \theta_1 - \beta'^2 \theta_2} \hat{K}_1 + \frac{(R - \tau) \hat{g}_0}{1 - \beta' \tau} \\ &\quad + \left(\beta' \left(\frac{1}{\eta} - \rho \right) + \rho \right) \left[\frac{\tau}{1 - \beta' \tau} - \frac{(\theta_0 \tau + \beta' \theta_2)}{\theta_0 - \beta' \theta_1 - \beta'^2 \theta_2} \right] A_\tau \hat{g}_0, \end{aligned}$$

which solves for \widehat{q}_0 as a function of \widehat{K}_1 . Also, the following equation relates \widehat{q}_0 , \widehat{q}_1 and \widehat{K}_1

$$\widehat{q}_0 = \beta' \widehat{q}_1 + (1 - \beta') \rho^\pi \widehat{K}_1 + (1 - \beta')(R - \tau) \widehat{g}_0.$$

E. Solution for $\pi > 0$

When the steady-state inflation is not zero, but $\pi > 0$, then the simple rule that following a money shock at $t = 0$ we can guarantee convergence of d_t back to the steady state by imposing $\tau < 1$ does not hold anymore. Recall that since $H_t^s = \tau H_{t-1}^s$ and when $\pi = 0$ we have $H^s = 0$, then $\tau < 1$ is enough to guarantee that H_t^s eventually converges to zero. In contrast, this is not the case when $\pi > 0$ then $d > 0$. Thus, when $\pi > 0$ the “sterilization” policy needs to be changed.

In particular, assume that the economy starts off the steady state and at time $t = 0$ there is an unexpected increase in growth rate of money $\varepsilon > 0$, i.e., $\widehat{g}_0 = \frac{\varepsilon}{1+\pi}$. In this case, the government chooses a period $t = T$ such that from T on, the growth rate of money supply is zero, i.e., $\widehat{g}_t = 0$ for $t \geq T$. What this implies is that for $t \geq T$, the law of motion of \widehat{d}_t is given by³⁴

$$\widehat{d}_t = \frac{1}{\beta'} \widehat{R}_t + \frac{1}{\beta'} \widehat{d}_{t-1},$$

which is clearly unstable, since $\beta' < 1$. Iterating forward on the equation above and imposing the transversality condition that $\widehat{d}_\infty = 0$, we obtain that \widehat{d}_{T-1} must satisfy:

$$\widehat{d}_{T-1} = - \sum_{\tau=0}^{\infty} \beta'^\tau \widehat{R}_{\tau+T},$$

to guarantee convergence back to the steady-state. Further, since using the law of motion of money supply we have that \widehat{g}_{T-1} is given by

$$\widehat{g}_{T-1} = - \frac{d}{1+d} \widehat{d}_{T-1} + \frac{d}{\beta'(1+d)} \widehat{R}_{T-1} + \frac{d}{\beta'(1+d)} \widehat{d}_{T-2},$$

³⁴This equation is the linearized version of the law of motion of the money supply when $\widehat{g}_t = 0$.

so that \widehat{g}_{T-1} depends on \widehat{d}_{T-1} . In summary, when the government chooses a period T such that $\widehat{g}_T = 0$, it must also choose \widehat{g}_{T-1} to satisfy the transversality condition. Further for periods $1 \leq t < T - 2$ we allow the government to choose any exogenous law of motion for $\widehat{g}_t \leq 0$, i.e., any rule in which the monetary expansion at time $t = 0$ is reverted. For instance, a natural choice would be a gradual money contraction up to period $T - 2$ and a choice of \widehat{g}_{T-1} that satisfies the condition above.

When $\pi > 0$, the dynamics of capital are described by

$$\theta_0^\pi \widehat{K}_t = \theta_1^\pi \widehat{K}_{t-1} + \theta_2^\pi \widehat{K}_{t-2} + \frac{1}{1+\pi} \left[(1-2\beta') \widehat{g}_t + (1-r^h) \widehat{g}_{t-1} \right],$$

where

$$\begin{aligned} \theta_0^\pi &= 1 + (1-2\beta') \frac{\rho}{(1+\pi)}, \\ \theta_1^\pi &= \frac{(1-r^h)(1-\rho)}{(1+\pi)} + 1 + (1-2\beta') \frac{\rho}{(1+\pi)} - \frac{(1-\beta')}{\eta(1+\pi)}, \\ \theta_2^\pi &= -\frac{(1-r^h)(1-\rho)}{(1+\pi)}. \end{aligned}$$

Using the dynamic equation of capital, as well as the transversality condition for government debt, the law of motion of money supply and the forward-looking solution for capital prices it is possible to construct a system of 5 equations in 5 unknowns: \widehat{K}_{T-1} , \widehat{q}_{T-2} , \widehat{q}_{T-1} , \widehat{d}_{T-1} and \widehat{g}_{T-1} . Since this system is a function of past values \widehat{K}_{T-3} , \widehat{K}_{T-2} and \widehat{d}_{T-2} an iterative procedure that starts with a guess for \widehat{K}_1 must be implemented to find the solution.

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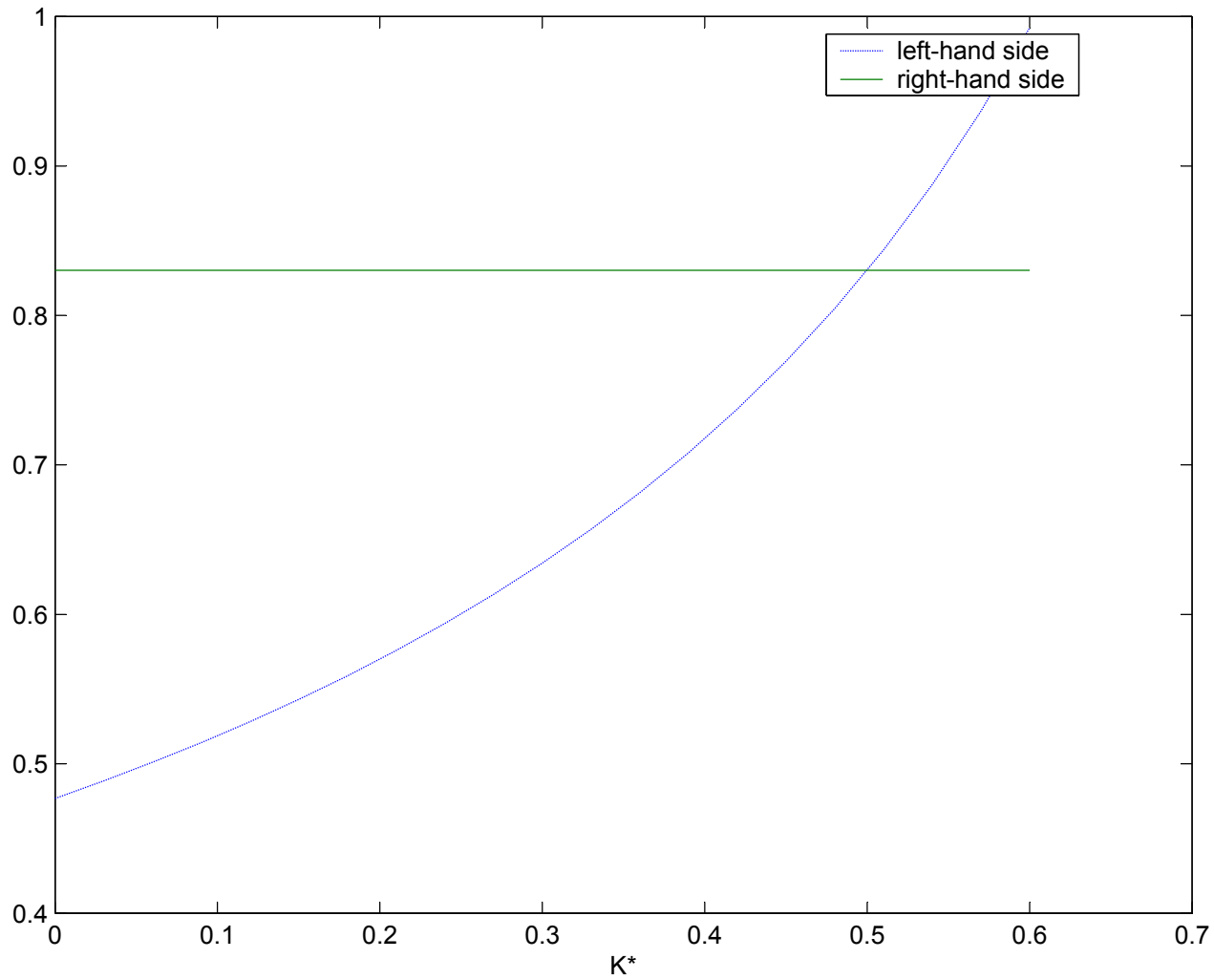


Figure 1: Steady state distribution of capital

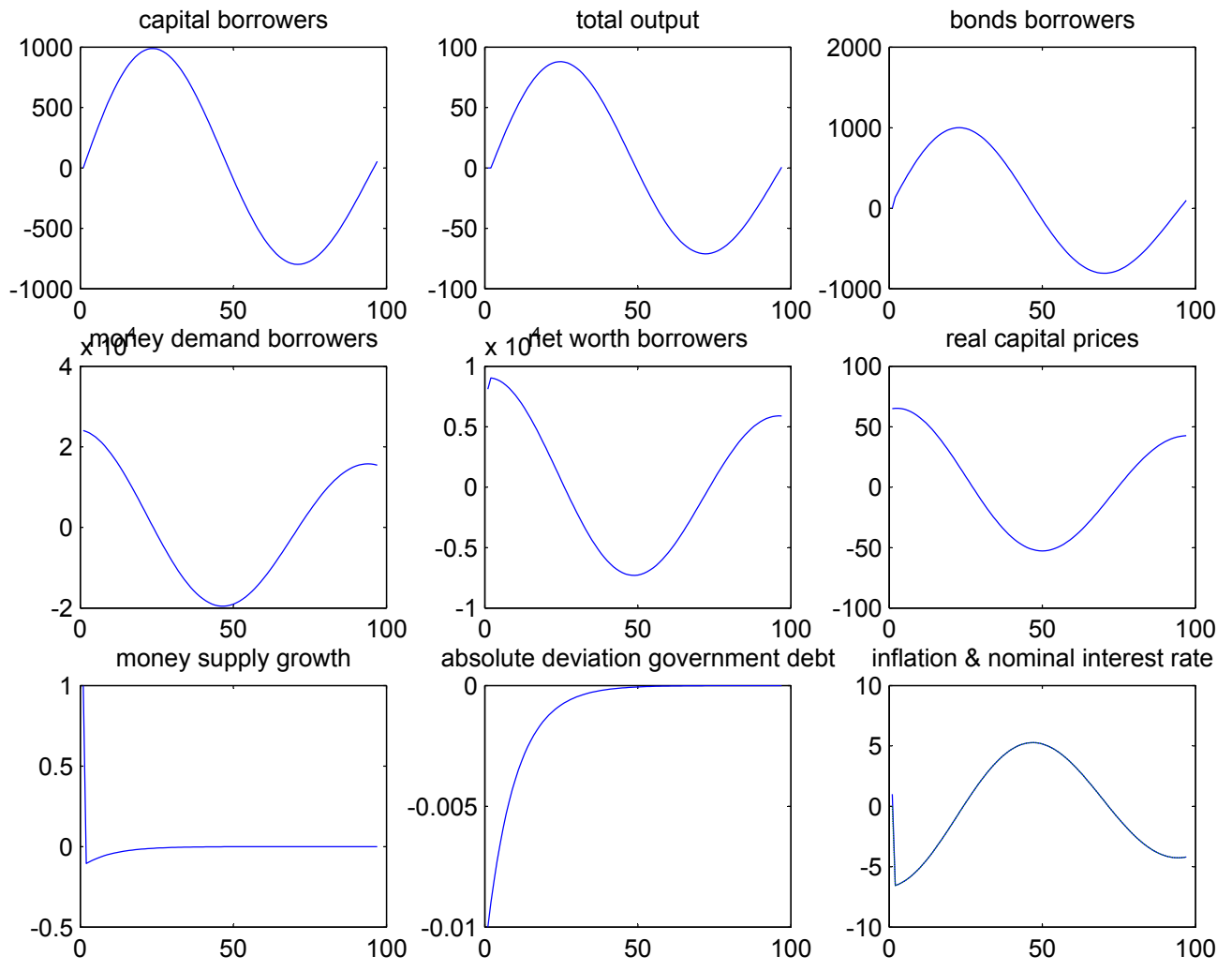


Figure 2: A 1% unexpected money expansion at t=0 with non-indexed debt

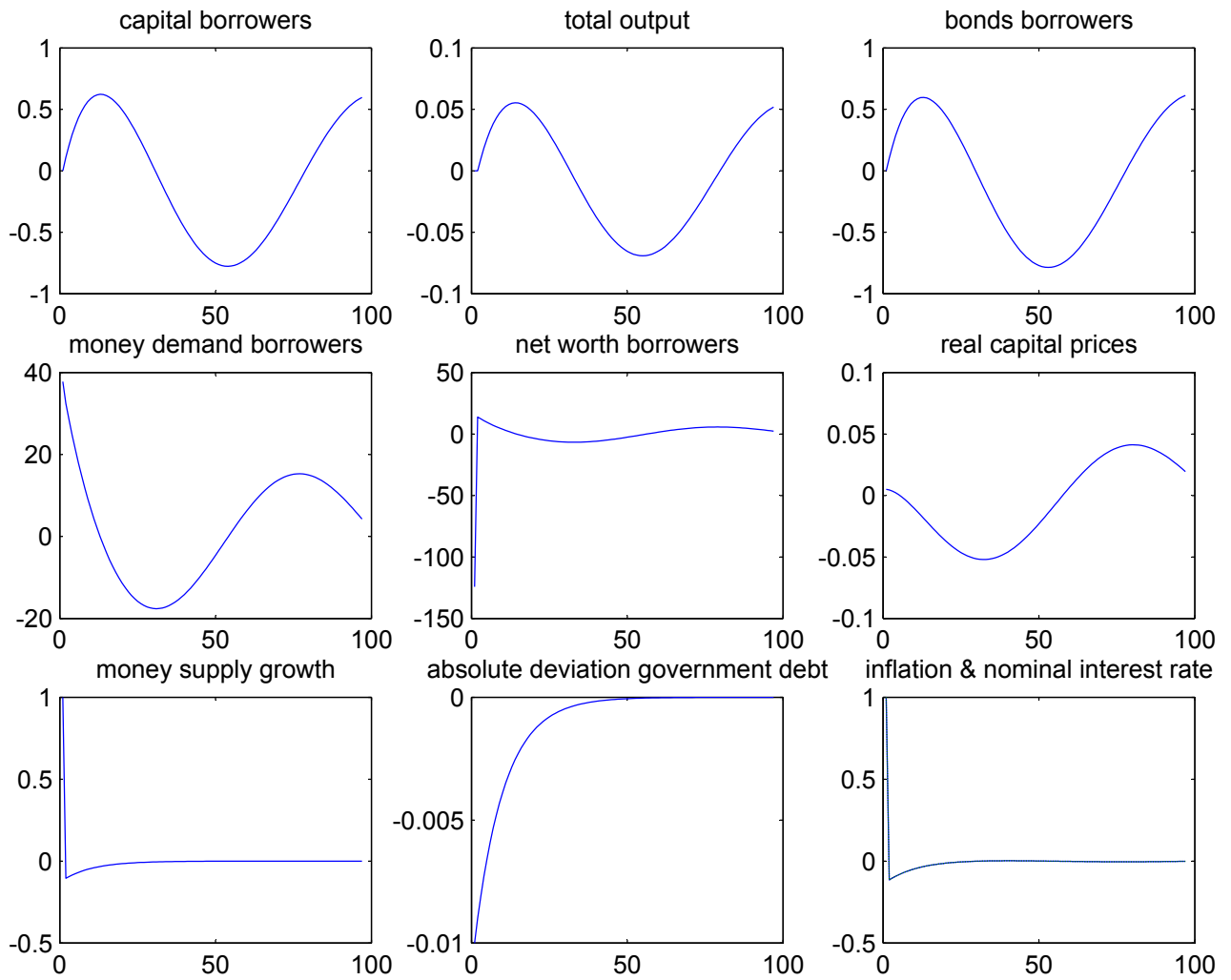


Figure 3: A 1% unexpected money expansion at t=0 with indexed debt

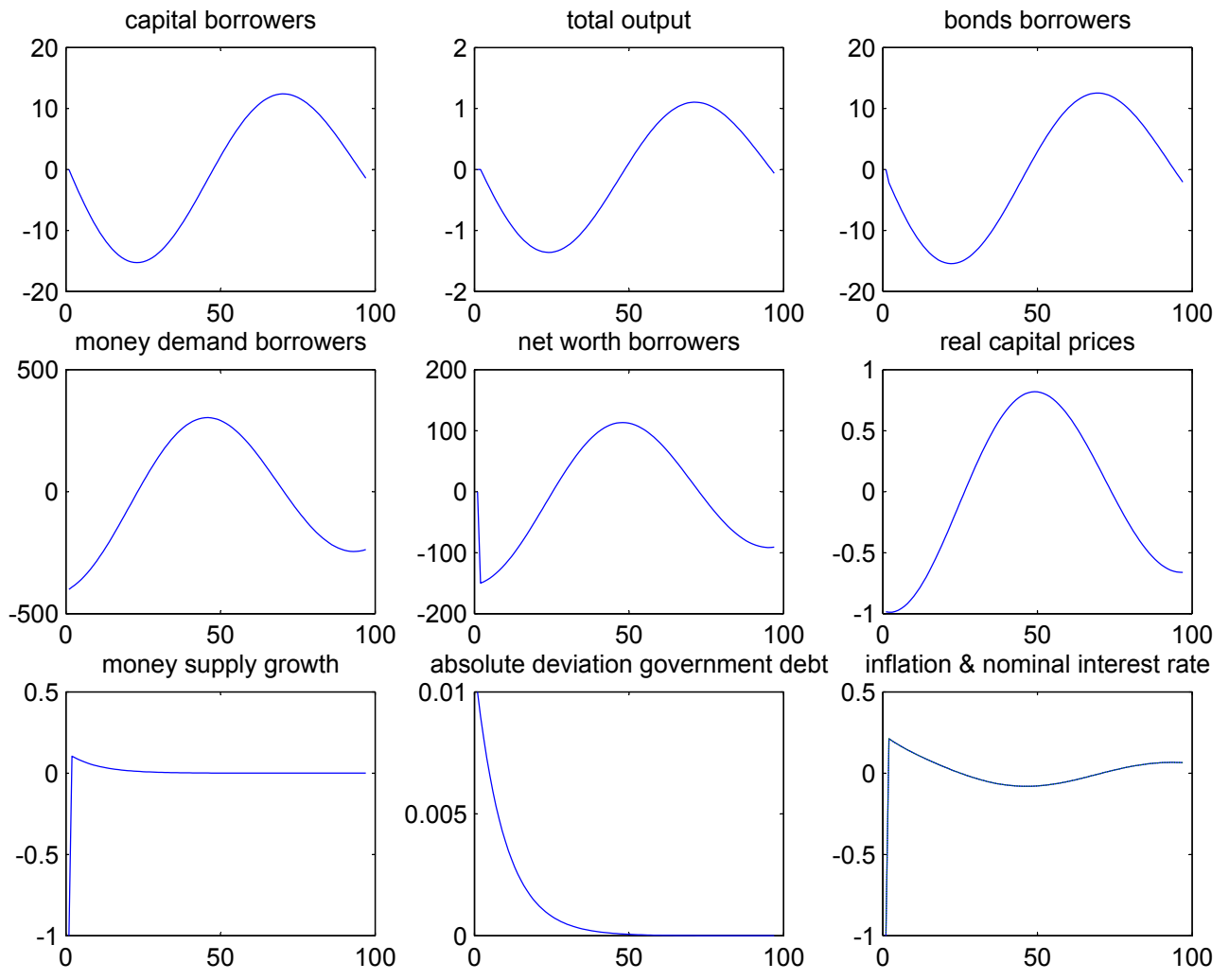


Figure 4: A 1% unexpected money contraction at t=0 with debt renegotiation