A Contribution to the Economic Theory of Fertility*

Juan Carlos Córdoba[†]

and

Marla Ripoll[‡]

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Abstract

Altruistic individuals regard fertility as a form of longevity. As a result, standard time-separable altruistic models of the type commonly used in macroeconomics cannot account for the evidence of increasing longevity but decreasing fertility as income rises, a puzzle. We show that a non-separable formulation of preferences that allows for a low elasticity of intertemporal substitution (EIS) but a high elasticity of intergenerational substitution (EGS) can explain the longevity-fertility puzzle. The model with a single elasticity cannot account for both. Our results suggests a major role for a new parameter in macro, the EGS. While the EIS mostly influences short-term economic decisions, the EGS influences mostly long-term economic choices.

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[†]Department of Economics, Iowa State University, 279 Heady Hall, Ames IA 50010, e-mail: cordoba@iastate.edu.

[‡]Department of Economics, University of Pittsburgh, 4933 W.W. Posvar Hall, Pittsburgh PA 15260, e-mail: ripoll@pitt.edu.

1 Introduction

The evidence on longevity and fertility is puzzling. On the one hand, longevity behaves like a superior good because both life expectancy and the share of health expenditures are positively related to income. On the other hand, fertility behaves more like an inferior good as it seems to fall with income. Figure 1 illustrates both facts for a cross-section of countries. The evidence is puzzling from the standpoint of dynamic altruistic models of the type commonly used in macroeconomics. The reason is that individuals who are altruistic toward their children should regard fertility as a form of longevity, just as an alternative way of living longer, and therefore the economic forces driving the increasing demand for longevity should, at least in principle, also lead to an increasing demand for fertility. This argument is formalized in a companion paper (Cordoba and Ripoll, 2011) that uses a version of the Barro-Becker model of fertility to show that the mechanism proposed by Hall and Jones (2007) to explain the increasing demand for longevity also results in an increasing demand for fertility.[1] A challenge for altruistic models is to account for the join longevity-fertility evidence, and in particular for the negative fertility-income relationship. As Jones, Schoonbroodt and Tertilt (2008) emphasize in their conclusion, it is hard to rationalize this negative relationship using dynamic altruistic models.[2]

The idea that fertility declines with income has a long tradition in economics. It was the dominant view held by many countries during the first conference on population sponsored by the United Nations, the 1974 World Population Conference, a view that was summarized at time by the adage "development is the best contraceptive." Evidence on a negative association is clear when looking at the cross-country data on average fertility and per-capita income, as illustrated in Figure 1. Similar evidence is obtained when looking at a cross-section of individuals within a country. For example, Becker (1960) finds a negative fertility-income relationship in the 1910, 1940 and 1950 Censuses and the Indianapolis survey for the 1900s, a pattern confirmed by Jones and Tertilt (2006) using US Census evidence dating back to 1826. Influential economic theories, such as those of Kremer (1993) or Hansen and Prescott (2002), endogenize population dynamics using simple rules that guarantee fertility falls with income at high income levels.

This paper develops a general dynamic theory of fertility choices by altruistic parents and applies the theory to provide an explanation for the longevity-fertility puzzle. Our key contribution is to formalize the concept of altruism from a basic set of axioms and to derive a general class of altruistic preferences suitable for dynamic macroeconomic models. We then use a version of our preferences to provide an explanation for the longevity-fertility puzzle. The class of preferences we derive includes as a special case the standard separable representation used by Barro and Becker (1989) and Becker and Barro (1988) (BB henceforth), but more importantly, it also includes non-separable representations of the Epstein-Zin (1989) type as well as many others. To the extent of our knowledge, fertility models with non-separable preferences have not been studied before.

[1] In Section 2 we summarize the results of Cordoba and Ripoll (2011).

[2] Their final conclusion reads: "expanding the successful [fertility] models to full dynamic versions based on parental altruism is very challenging. Dynamic models are very important for understanding the connection between cross-sectional fertility differences and the demographic transition. More research in this area is needed" (p. 60)

Providing microfoundations of the demand for children is important for internal consistency. We discuss examples of seemingly altruistic preferences used in the literature that violate simple altruistic principles, such as indifference. In our formulation, the fundamental axiom of altruism is that the utility of the parent increases with the utility of each one of their "potential" children. An implication of this axiom is that it requires the explicit consideration of the utility of unborn children. Most of the fertility literature does not explicitly consider the utility of children in the unborn state because it is implicitly normalized to zero. However, we show that in the context of non-separable preferences such normalization is not without loss of generality. The microfoundations for dynamic altruistic preferences we provide have a broader impact on problems beyond those of fertility choices. For example, any decision problem in which altruistic individuals must choose between two alternative states with different utility values such as giving birth to a child or not, living or dying, staying married or getting divorced, should satisfy an axiom of indifference. Specifically, if the two alternatives have the same utility value, the individual should be indifferent among them. In this paper we postulate a number of axioms that discipline the class of admissible preferences consistent with altruism.

We use our non-separable altruistic preferences to study fertility choices in a life-cycle model with a non-negative bequest constraint.[3] A key feature of our preferences is that they can disentangle the curvature associated to intragenerational consumption from that of intergenerational consumption choices. That is, they disentangle what we call the "elasticity of intergenerational substitution" (EGS) from the elasticity of intertemporal substitution (EIS). This disentangling captures a dimension of intergenerational consumption allocation that, to the extent of our knowledge, has not been explored before. Standard time-separable preferences implicitly assume that the rate at which the parent substitutes his own consumption across time is the same rate at which the parent substitutes his own and his children's consumption. This does not need to be the case.

The distinction between the EGS and the EIS is key to resolve the fertility-longevity puzzle. On the one hand, it allows to assume $EIS < 1$, which is not only standard in quantitative macro, but also a restriction required by Hall and Jones (2007) to explain why longevity is a superior good. In their separable context, a low EIS implies a strongly diminishing marginal utility of income, while the marginal benefit of life extensions remains bounded. This feature of preferences explains why richer individuals want to spend an increasing fraction of their income in health in order to prolong their life span. On the other hand, we show that fertility decisions depend on the EGS. We find that regardless of the EIS, if $EGS > 1$ then fertility decreases with wages. The standard separable case restricts $EIS = EGS$ and therefore cannot simultaneously account for the longevity and fertility evidence, while our non-separable case can because it allows for $EIS < 1$ and $EGS > 1$. Since the EGS is a new concept, no estimates of its value exist but we consider the fact that fertility decreases with measures of income as prima facie evidence that $EGS > 1$ and leave its estimation for future work.

The intuition for why $EGS > 1$ is needed to obtain a negative fertility-income relationship is the

[3] Bequest constraints are a form of financial frictions. Cordoba and Ripoll (2011) show that other financial frictions, such as borrowing constraints for students, have similar implications.

following. Suppose wages go to infinite and that the $EGS < 1$. Parents drive their consumption to infinite as part of the optimal plan. Moreover, the low elasticity of substitution between children's and parental consumption induces the parent to provide an increasing consumption to their children. More importantly, providing consumption to a new child becomes increasingly more valuable than rising consumption of existing family members. For this reason, parents facing a very large wage would like to have as many children as possible if $EGS < 1$. In contrast, if $EGS > 1$ then parental consumption can substitute for children's consumption, and the consumption of newborns is not particularly valuable relative to the consumption of the parent. In this case, the number of children does not need to increase with wages and in fact, as we show, decreases with wages in the presence of non-labor income or non-homothetic preferences. In sum, the main insight of our analysis is that as income grows both demand for longevity and children rise under time-separable preferences. In contrast, non-separable preferences that allow for $EIS < 1$ and $EGS > 1$ can account for an increasing demand for longevity and falling demand for children. In this case, as income raises, individuals value extending their own life, but they do not value newborns as much as their own consumption.

A large literature has sought to explain the evidence on the fertility-income relationship. The most accepted explanation is the time-cost of children theory according to which the opportunity cost of having children is higher for high earning individuals (Barro, 1960; and Barro and Lewis, 1973). Such theories are able to explain the required pattern by including non-labor income and/or non-homothetic preferences. However, preferences in this early tradition are mostly static, and as Jones, Schoonbroodt and Tertilt (2008) point out in their review paper, the results obtained with these static preferences do not seem to hold under fully dynamic altruistic preferences, and call for more research in this area. Dynamic altruistic fertility models in the tradition of BB include Becker, Murphy and Tamura (1990), Alvarez (1999), Boldrin and Jones (2002), Barro and Sala-i-Martin (2004), Doepke (2004, 2005), Manuelli and Seshadri (2009), Jones and Schoonbroodt (2009, 2010) and Bar and Leukhina (2010), among others. A key feature of the BB model, the workhorse in all this literature, is that it predicts that fertility is independent of wages but it depends on the interest rate. The reason why fertility is independent of wages is that bequests are unconstrained. Although interest rates may play a role in explaining cross-country fertility differences (Manuelli and Seshadri, 2009), this mechanism may not be relevant to explain fertility differences within a country. Differences in wages may be a more plausible explanation.

In a companion paper, Cordoba and Ripoll (2011) introduce financial frictions in the BB model and show that in this case wages become a determinant of fertility. However, in this constrained BB model, obtaining a negative fertility-income relationship with time-separable CRRA preferences requires $EIS > 1$. But this is problematic because as discussed before, $EIS < 1$ is the value supported by ample empirical evidence, and the value consistent with the observed positive relationship between income and demand for longevity (Hall and Jones, 2007). These results motivated us to look beyond the standard separable preferences.

A parallel and complementary literature in macroeconomics has studied fertility in non-altruistic settings (i.e., those in which parents care about either the number of children, or their human capital, but not directly about the utility of the children). Examples in this category of papers include Galor and Weil (1996, 1999, 2000), Greeenwood and Seshadri (2002), Hansen and Prescott (2002), Greenwood, Seshadri and Vandenbroucke (2005), and Galor (2005, 2011), among others. Our paper focuses instead on dynamic altruistic preferences in the tradition of Barro and Becker. These preferences are widely used in macroeconomics but, as discussed before, are not able to simultaneously account for the longevity-income and the fertility-income relationships observed in the data. Our paper seeks to resolve this issue. The result of this effort is the formulation of a general class of altruistic preferences that are useful in studying fertility choices. Our model generates a negative link between fertility and wages, one that allows it to have predictions both for the time series and the cross-section. Moreover, another contribution of our paper is that in formulating non-separable altruistic preferences we propose the notion of intergenerational substitution as a separate one from intertemporal substitution. The intergenerational substitution notion may be useful in studying a variety of other relevant questions in macroeconomics that involve allocation of resources across generations.

The remainder of the paper is organized as follows. Section 2 provides a preamble by summarizing the findings of dynamic altruistic fertility choice models with standard time-separable preferences. Section 3 derives the relevant altruistic preferences from first principles or axioms. The result of our axiomatic approach is a generalized set of internally consistent altruistic preferences that can be used to derive the demand for children. In Section 3.6 we provide specific examples of a variety of altruistic preferences that can be easily incorporated in dynamic general equilibrium models. In particular, we consider functions of the CRRA and CARA families, and stress a formulation that disentangles the EIS, the one that controls consumption smoothing within a generation, from the EGS, which controls consumption smoothing across generations. Section 4 studies fertility decisions of altruistic parents within a dynamic set up. Parents face time and non-time costs of raising children. Bequests are in principle possible, but we focus on cases in which the interest rate is sufficiently low so that the bequest constraint is binding. The resulting set up resembles the Samuelson-Diamond OLG model but with endogenous dynasties. In Section 5 we verify that our fertility choice model preserves the prediction in Hall and Jones (2007) that the demand for longevity increases with income. Finally, Section 6 concludes.

2 The separable case

Consider the following fertility choice problem solved by altruistic parents in a life-cycle economy. A parent with life span T chooses a life cycle consumption profile $C = [c_0, c_T]$, the number of children $0 \leq n \leq N$, and bequests b' in order to maximize lifetime utility subject to a present value budget constraint and a non-negative bequest constraint. The parent is assumed to have all children at age F.[4] The parental problem is described by the following Bellman equation:

$$V(b) = \max_{C=[c_0,c_T],0\leq n\leq N,b'} \left[U + \alpha\Phi(n)V(b')\right] \tag{1}$$

subject to

$$b + w\textstyle\int_0^T e^{-rt} l_t(n)dt + Y = \int_0^T e^{-rt} c_t dt + n\left(\chi + e^{-rF}b'\right) \tag{2}$$

$$b' \geq 0. \tag{3}$$

Function U represents the parent's personal utility as given by

$$U = \textstyle\int_0^T e^{-\rho t}\frac{c_t^{1-\sigma}}{1-\sigma}dt + \underline{U}$$

with $\sigma > 0$ and $\underline{U}$ is a constant. This representation of U corresponds to the standard time-separable CRRA utility, where $1/\sigma$ is the EIS. The only difference here is that we introduce $\underline{U}$ in order to ensure $U \geq 0$. Specifically, the presence of $\underline{U}$ allows to consider the case $\sigma > 1$, one that is standard in the growth and business cycle literature. The original BB framework restricts attention to the case $0 < \sigma < 1$, one in which $\underline{U}$ can be normalized to 0.

Term $\alpha\Phi(n)V(b')$ in (1) represents the total utility of the n children, where α is the "degree of altruism" and function $\Phi(n)$ is increasing and concave (i.e., $\Phi'(n) > 0$ and $\Phi''(n) < 0$). Concavity captures diminishing marginal altruism. Term $\alpha\Phi(n)$ is then the weight the utility of each child $V(b')$ has on the parent's utility.

Turning now to the present value budget constraint in (2), let the parental lifetime labor supply be $L(n) \equiv \int_0^T e^{-rt} l_t(n)dt$, where r is the instantaneous interest rate and $l_t(n)$ is the labor supplied at age t. Assume that $L'(n) < 0$ so that children are costly because they reduce the parent's labor supply. We can then define the parental lifetime income net of children costs as $I(n) \equiv wL(n) + Y - n\chi$, where Y is the present value of non-labor income and χ is the non-time costs of raising each child. Non-labor income has been part of earlier static fertility choice models. It is interpreted as gifts, lottery income, or any income that is not tied to labor. Alternatively, in two-parent fertility models, Y has been interpreted as the husband's income, case in which w is the wife's wage. Finally, equation (3) corresponds to the non-negative bequest constraint.

The original BB parental problem is a special case of the one described above. In addition to restricting attention to the case $0 < \sigma < 1$, BB do not require bequests to be non-negative. This has important implications for the determinants of fertility. Specifically, absent constraint (3) the optimality condition for bequests is given by

$$\frac{c_0'}{c_0} = \left[\frac{\alpha\Phi(n)e^{rF}}{n}\right]^{1/\sigma}$$

[4] The separable model discussed in this section draws from our companion paper Cordoba and Ripoll (2011).

where c_0'/c_0 is the child-to-parent relative initial consumption. In the steady state of the model, this equation determines optimal fertility n^* from

$$\frac{\alpha\Phi(n^*)}{n^*} = e^{-rF}$$

so fertility is a function of the interest rate, but it is independent of all level variables w, Y, and I. In sum, in BB fertility is independent of income (wages). This stems from the fact that absent a non-negative bequest constraint, steady-state fertility is determined by the bequest optimality condition, and not by the fertility optimality condition.

In contrast, if the non-negative bequest constraint (3) is binding, the optimality condition for fertility is given by

$$\left(-\frac{\partial I(n)}{\partial n}\right) \times \frac{\partial U^*}{\partial I(n)} = \alpha\Phi'(n)V(b') \tag{4}$$

where the left-hand side is the marginal cost and the right-hand side is the marginal benefit of an additional child both measured in personal utils (or composite good). Before discussing the details of how U^* is obtained, briefly consider the effects of wages on the marginal benefits and costs. The marginal benefit of children increases with wages because it increases the welfare of children $V(b')$, which raises the welfare of the parent as well. Regarding the marginal cost, there are two opposing effects: the opportunity cost of foregone labor income $(-\partial I(n)/\partial n)$ increases with w for any given number of children, while the marginal utility of income $\partial U^*/\partial I(n)$ decreases with w. For fertility to be a negative function of income, it must be the case that in response to higher w the marginal cost increases by more than the marginal benefit. This requires the increase in the opportunity cost of foregone labor income to be strong enough to offset the decrease in the marginal utility of income.

Turning to more specific details of equation (4), U^* corresponds to utility under the optimal consumption path for given n and $b = b' = 0$. In other words, U^* solves the following subproblem

$$U^*\left(I\left(n\right)\right) = \max_{C=[c_0,c_T]} U(C) \text{ subject to } I(n) \geq \textstyle\int_0^T e^{-rt}c_t dt$$

which results in

$$U^*(I(n)) = \frac{\Theta}{1-\sigma} I(n)^{1-\sigma} + \underline{U}$$

where $\Theta > 0$ collects constants.[5] The marginal utility of income in this case is given by $\partial U^*/\partial I(n) = \Theta I(n)^{-\sigma}$, which implies that the smaller the σ, the smaller the decrease in the marginal utility of income in response to higher w, and the more likely a negative fertility-income relationship would hold. Using the definition of $I(n)$ together with the expression for $\partial U^*/\partial I(n)$ and the stationary

[5] Specifically, Θ is given by:

$$\Theta = \left(\frac{r-(r-\rho)/\sigma}{1-e^{((r-\rho)/\sigma-r)T}}\right)^{1-\sigma} \left(\frac{1-e^{((r-\rho)(1-\sigma)/\sigma-\rho)T}}{\rho-(r-\rho)(1-\sigma)/\sigma}\right) > 0.$$

value of $V(b)$ we have can write equation (4) as[6]

$$\big(-wL'(n)\big) \times \left[\Theta(wL(n)+Y)^{-\sigma}\right] = \alpha\Phi'(n)\frac{\frac{\Theta}{1-\sigma}(wL(n)+Y)^{1-\sigma}+\underline{U}}{1-\alpha\Phi(n)}. \tag{5}$$

The equation above implies that if $Y = \underline{U} = 0$, then fertility is independent of income. In other words, it is possible to have wages w as a determinant of fertility when the non-negative bequest constraint binds, but only in the presence of non-labor income or non-homotheticity. To further understand the role of non-labor income Y in (5), consider the case $\underline{U} = 0$, which would only allow to consider $\sigma < 1$. In this case, with $Y > 0$ an increase in w increases the marginal cost proportionally more than it increases the marginal benefit, so fertility declines with w. The increase in the marginal cost (left-hand-side of equation 5) is larger due to two reasons: first, it involves a term linear in w, the opportunity cost $-wL'(n)$; and second, the decrease of the marginal utility of income $\Theta(wL(n)+Y)^{-\sigma}$ when w rises is weakened by the presence of Y. At the same time, the increase in the marginal benefit (right-hand-side of equation 5) through term $(wL(n)+Y)^{1-\sigma}$ is weakened by the presence of Y.

To understand the role of $\underline{U}$ in (5), consider the case $Y = 0$, so that we can write (5) as

$$\frac{-L'(n)}{L(n)}\Theta\left(wL(n)\right)^{1-\sigma} = \alpha\Phi'(n)\frac{\frac{\Theta}{1-\sigma}\left(wL(n)\right)^{1-\sigma}+\underline{U}}{1-\alpha\Phi(n)}$$

which implies that $\underline{U} > 0$ weakens the effects of higher w on the marginal benefit of children. Specifically, if $\sigma < 1$ an increase in w increases the marginal benefit less than it increases the marginal cost, and thus fertility is a negative function of income. However, if $\sigma > 1$, the marginal cost decreases and fertility increases with income. In sum, provided that either $Y > 0$ or $\underline{U} > 0$, fertility is a negative function of income only when $0 < \sigma < 1$.

The discussion above hints at a key characteristic of fully altruistic dynamic models. Since in these models parents care about the utility of the children, an increase in w directly increases the marginal benefit of having children through the increase in the utility of the chid $V(b')$. In this context, a negative fertility-income relationship can only be obtained if the positive effect of w on the marginal benefit of a child can be weakened. This is the role Y and $\underline{U}$ play here: if there is more income than just labor income, or if there is more utility in life that the one derived from purchasing consumption goods, then higher wages should have a relatively weaker effect in increasing the marginal benefit of having children. Effectively speaking, both Y and $\underline{U}$ act as a built-in non-homotheticity in preferences. However, we still have an issue here. The equations above suggest that dynamic altruistic models of fertility with standard CRRA time-separable preferences, binding non-negative bequest constraints and non-labor income are able to predict a negative fertility-income relationship only when $EIS > 1$. This is problematic because

[6] In deriving this equation we have assumed $\chi = 0$, so that the only cost of raising children is in terms of the parent's time. As Jones, Schoonbroodt and Tertilt (2008) discuss, this is the most relevant cost in understanding the fertility-income relationship.

$EIS < 1$ is the value supported by ample empirical evidence, and it is the standard value in quantitative growth and business cycle models. In addition, as shown in Hall and Jones (1997), $EIS < 1$ is required to explain the increased demand for longevity as income increases in time: when $EIS < 1$ life is a superior good.

In what follows we propose the introduction of non-separable utility to study models of fertility choice. In particular, we propose to replace (1) with

$$V(b) = \max_{C=[c_0,c_T],0\leq n\leq N,b'} G(U(C), V'(b), n) \tag{6}$$

where function G is not necessarily linear, and function $U(C)$ is non-separable. One of the advantages of this general representation is that it allows to disentangle the EIS from what we call the "elasticity of intergenerational substitution" (EGS). One example of the class of non-separable preferences $G(U, V', n)$ is given by

$$V = \left[U^{1-\eta} + \alpha\Phi(n)V'^{1-\eta}\right]^{\frac{1}{1-\eta}} \tag{7}$$

with

$$U = \left(\int_0^T e^{-\rho t} c_t^{1-\sigma} dt\right)^{\frac{1}{1-\sigma}} + \underline{U}$$

where $1/\sigma$ is the EIS, while the EGS is given by $1/\eta$. Notice that when $\underline{U} = 0$, the separable representation can be obtained as a special case when $\sigma = \eta$. As we show below, in our non-separable framework a negative fertility-income relationship can be generated with a low EIS ($\sigma > 1$), and a high EGS ($\eta < 1$). In this case, parents prefer flatter consumption profiles as in standard macro models ($\sigma > 1$), but they also have a high degree of substitution between parental and children's consumption ($\eta < 1$). This implies that in the face of higher income, the consumption of an additional child is not particularly valuable relative to the consumption of the parent. Under these circumstances, a negative fertility-income relationship can be obtained with $\sigma > 1$.

These more general, non-separable altruistic preferences have not been studied before. Next section we derive these preferences from first principles by proposing a set of axioms and providing examples of utility functional forms that satisfy them. It turns out, as we describe in detail below, that providing these microfoundations for altruistic preferences is important in understanding the underlying principles of demand for children. We view altruism as a natural explanation for the demand for children. It implies that the parent is the "social planner" at the family level so that family allocations are not intrinsically inefficient. After all, altruism provides microfoundations to the household's preferences in the Ramsey model, a central model in modern macroeconomics.

3 The altruistic welfare function

In this section we postulate a set of axioms describing preferences of altruistic parents and derive certain restrictions and properties implied by these axioms. We also provide examples of preferences that satisfy the axioms and examples of preferences, some of which are used in the literature, that do not.

3.1 The altruistic approach

Consider a continuous time dynastic set up in which individuals derive utility from their private consumption and from the utility of their N potential children. Among the potential children, only n are born where $0 \leq n \leq N$. Potential children are ordered according to their "potential" birth ordering, from 1 to N. The birth ordering is "potential" because children may or may not be born. Assume that parental utility, V, can be written in terms of a private utility index associated to his own consumption, U, and the utility of his N potential children, V^1 to V^N, as follows:

$$V = \widehat{G}(U, V^1, V^2, .., V^N). \tag{8}$$

This formulation is similar to the one used by Koopmans (1960), Lucas and Stokey (1983), Dolmas (1996), Becker and Boyd (1997), Ben-Gad (1998), Backus, Routledge, and Zin (2004), and Farmer and Lahiri (2005), but with three important differences. First, U is an utility index, or composite consumption good, associated to the lifetime consumption sequence of the parent, not just the consumption of one period. Second, the formulation above allows for multiple potential descendants N instead of just one. Third, we study the endogenous determination of born descendants n. In what follow we use the terms born descendants, children and offsprings interchangeably.

Function (8) is altruistic because the parent cares about the welfare of his descendants. It may not be immediately intuitive why the welfare function should specify the utility of all potential children, even if they are unborn. The following example seeks to motivate this important point.

Example 1. *Suppose $N = 1$ and $\widehat{G}$ is increasing in its two arguments. Let $V^1 = V'$ be the utility of the descendant if born and $V^1 = D$ if unborn. Similarly, let U_0 and U_1 be the personal utility associated to having zero or one child respectively. Suppose $U_0 > U_1$ meaning that the child is costly to the parent in terms of personal utility. The child is born if $\widehat{G}(U_1, V') > \widehat{G}(U_0, D)$ which requires $V' > D$.*

The previous example makes clear that altruistic models need to specify the utility of the child in the unborn state, D. Since the parent cares about another individual, his child, the utility of that individual if unborn is relevant for fertility decision of the parent. Most of the fertility literature does not explicitly consider the utility of the unborn either because it is implicitly normalized to zero, or because the underlying preferences are not truly altruistic.[7] The welfare of the unborn

[7] An exception is Jones and Schoonbroot (2009). See Example 3 below for a discussion.

is also explicitly considered in models of optimal population size. For example, the question of optimal population size requires the concept of "potential people" and their welfare in the born and unborn states (see Golosov, Jones and Tertilt, 2007). Appendix A develops an alternative motivation for D by considering the equivalence between children and longevity.

The utility function (8) resembles the Dixit and Stiglitz's (1977) preferences for varieties. In their formulation U is the consumption of a numeraire good while $(V^1, V^2, .., V^N)$ are the consumptions of a range of "potential" goods. Moreover, some goods are not consumed in equilibrium meaning that for those goods $V^i = 0$. In our framework, parental utility depends on the utility of the N "potential" children some of whom are not born, meaning that $V^i = D$ for those children. In the Dixit-Stiglitz's preferences individuals derive utility from a "variety" of potential consumption goods, while in our preferences parents derive utility from a "variety" of potential children.

Function $\widehat{G}$ is an aggregator. A simplification arises when all born children receive the same utility, say V', and all unborn children receive the same utility, D. This situation could arise due to properties of the preferences or to underlying technological or social constraints.[8] This is the case stressed in the literature and the one we focus in most of the paper. In this case (8) simplifies to

$$V = G(U, V', D, n) \equiv \widehat{G}(U, \mathbf{V}'_n, \mathbf{D}_{N-n}), \tag{9}$$

where $\mathbf{V}'_n = V' \times \overrightarrow{\mathbf{1}}_n$ and $\mathbf{D}_{N-n} = D \times \overrightarrow{\mathbf{1}}_{N-n}$, with $\overrightarrow{\mathbf{1}}_m$ a m dimensional row vector of ones.

We now postulate some desirable properties for $\widehat{G}$ in the form of axioms, and use corollaries to translate those axioms into implied properties for the function G. Of particular importance is to characterize the derivative of G with respect to n, key in determining fertility choices. Notice that n does not enter directly as an argument in the primitive function $\widehat{G}$ but enters indirectly in function G because it divides the set of potential children into two groups, born and unborn, which in turn enjoy different utility levels.

It is natural to restrict the number of children to be a discrete variable. On the other hand, it is convenient to assume that the number of children is a continuous variable so that simple calculus can be used to characterize fertility decisions. With these considerations in mind, we state the axioms characterizing $\widehat{G}$ for a discrete number of children but state the implied properties of G for a continuous number of potential children taking values in the interval $[0, N]$.

3.2 Basic axioms of altruism

Let $U^F \times V^F$ be the set of feasible utilities U and V. Assume that $\widehat{G}$ is differentiable as needed in the feasible set and that unborn children all receive the same utility, $D \in V^F$. Denote the partial derivatives of $\widehat{G}$ as $\widehat{G}_U \equiv \partial\widehat{G}/\partial U$ and $\widehat{G}_i \equiv \partial\widehat{G}/\partial V^i$ for $i = 1, ..., N$.[9] A natural property is $\widehat{G}_U > 0$ which is assumed to hold. Similarly, denote $G_U \equiv \partial G/\partial U$, $G_V \equiv \partial G/\partial V'$, and $G_D \equiv \partial G/\partial D$. A key concept is the increase in parental utility derived from one more offspring. It can be defined as $G(U, V', D, n+1) - G(U, V', D, n)$. When n is allowed to be a continuos variable, the analogous

[8] For example, parents may equalize utilities across children to avoid conflicts among them.

[9] For a continuous number of children, $\widehat{G}_i$ is defined for $i \in [0, N]$ and the first child is child "0".

concept is that of the marginal utility of offsprings defined as $G_n(U, V', D, n) \equiv \partial G(U, V', D, n)/\partial n$. The following corollary establishes basic restrictions on G_U, G_V, G_D and G_n that follow directly from the definition of G given by (9).

Corollary 1. $G_U = \widehat{G}_U > 0$, $G_V = \int_0^n \widehat{G}_i(U, \mathbf{V}'_n, \mathbf{D}_{N-n})di$, $G_D = \int_n^N \widehat{G}_i(U, \mathbf{V}'_n, \mathbf{D}_{N-n})di$ and

$$G_n(U, V', D, n) \equiv \widehat{G}_n(U, \mathbf{V}'_n, \mathbf{D}_{N-n})(V' - D) \tag{10}$$

Equation (10) is important in understanding the nature of altruistic preferences.[10] It states that the marginal utility (to the parent) of having offspring n is the additional utility to the child from being born, $V' - D$, times the marginal effect of that utility into the parent's utility, $\widehat{G}_n$. We now turn to characterize these two terms: the following two axioms define $\widehat{G}$ as a pure aggregator of parents and children's utilities (Axiom 1), and define altruistic preferences as those for which $\widehat{G}_n > 0$ (Axiom 2). Next, Proposition 1 below pertains to term $V' - D$.

Axiom 1 - Identity. $\widehat{G}(U, ..) = U$ if $N = 0$.

Corollary 2. $G(U) = U$ if $N = 0$.

In words, Axiom 1 states that the utility of a parent with no potential children is just U. In that case, G is just the identity function. The next axiom defines altruism:

Axiom 2 - Altruism. $\widehat{G}_i > 0$ for all $i \in \{1, .., N\}$.

Corollary 3. Let Axiom 2 hold. Then $G_V > 0$ and $G_D > 0$.

Axiom 2 states that parental utility is strictly increasing in its last N arguments. Specifically, it states that under altruism, adding the n-th child has a marginal effect on the utility of the parent that is positive, i.e., $\widehat{G}_n > 0$. The following proposition refers to term $V' - D$, the additional utility for a child in going from the unborn to the born state.

Proposition 1 - Offspring are goods. Let Axiom 2 hold. Offsprings are goods (for the parents) if and only if $V^i > D$ for all $i \leq n$.

Proof. Offspring i is a good if, everything else equal, parental utility increases by having child i, or

$$\widehat{G}(U, V^1, ..., V^i, ..., V^n, D, D, ., D) - \widehat{G}(U, V^1, ..., D, ..., V^n, D, D, ., D) > 0$$

for all $i \leq n$. By Axiom 2, this inequality holds if and only if $V^i > D$.

Corollary 4. Let Axiom 2 hold. Then $G_n(U, V', D, n) > 0$ (offsprings are goods) if and only if $V' > D$ and $G_n(U, V', D, n) = 0$ (indifference) if and only if $V' = D$.

[10] Notice that equation (10) corresponds to the first order Taylor approximation of $G(U, V', D, n+1) - G(U, V', D, n) = \widehat{G}(U, \mathbf{V}'_{n+1}, \mathbf{D}_{N-n-1}) - \widehat{G}(U, \mathbf{V}'_n, \mathbf{D}_{N-n})$.

Equation (10) together with Axiom 2 and Proposition 1 are key in identifying whether any given preferences are altruistic or not. As we show at the end of this section, a number of altruistic preferences that have been used in the literature do not satisfy Axiom 2, so they are not consistent with Proposition 1 and Corollary 4 (see discussion after Proposition 2).

The degree of altruism toward potential child i is described by $\widehat{G}_i$ and the overall degree of altruism by $\sum_{i=1}^{N} \widehat{G}_i$. When $V_i' = V'$ for all offspring, the overall degree of altruism toward offspring is described by G_V. In order to have well-behaved fertility and bequest decisions, some restrictions are needed for both. A property commonly required in the literature is that of *diminishing altruism* or *child discounting.* It states that the marginal utility to the parent from additional offsprings decreases with the number of offsprings. This property is analogous to time discounting in dynamic models but applied to the number of children rather than the number of periods. The following is a formal definition of diminishing altruism.

Axiom 3 - Diminishing altruism or child discounting. $\widehat{G}_i(U, \mathbf{V}_n', \mathbf{D}_{N-n})$ strictly decreases with i for all $i \leq n$ and all feasible n. Moreover, $\widehat{G}_n(U, \mathbf{V}_n', \mathbf{D}_{N-n})$ decreases with n for all $n \leq N$.

Corollary 5. Let Axiom 3 hold. Then $G_n(U, V', D, n)$ decreases with n.

It is convenient to impose more structure to the rate at which altruism diminishes. The following axiom postulates a strong form of diminishing altruism used in the literature. It states that altruism decreases at a rate that only depends on the birth order but not on U, V, D or n.

Axiom 3a - Child discount factor. Let $\varphi(i)$ be a positive strictly decreasing function satisfying $\varphi(1) = 1$. Then, $\widehat{G}_i(U, \mathbf{V}_n', \mathbf{D}_{N-n}) = \widehat{G}_1(U, \mathbf{V}_1', \mathbf{D}_{N-1})\varphi(i)$ for all $i \leq n$ and all feasible n.

To further understand Axiom 3a, notice that $\varphi(i) = \widehat{G}_i(U, \mathbf{V}_n', \mathbf{D}_{N-n})/\widehat{G}_1(U, \mathbf{V}_1', \mathbf{D}_{N-1})$ is the "weight" of offspring i in a family with n children relative to that of offspring 1 in a family with one child. In principle, $\varphi(i)$ could depend on i, U, V, D and n, but Axiom 3a states that it only depends on i, the birth order. Variables such as family size or consumption do not affect this relative weight. This assumption is analogous to the assumption of standard dynastic models according to which the rate of time preference depends only on the time of consumption but not on other characteristics such as life span or consumption level.

Consider next some implications for G. Axiom 3a and equation (10) imply that $G_n(U, V', D, n) = G_0(U, V', D, 0)\varphi(n)$, which again assumes that n is a continuos variable in the interval $[0, N]$. Define $\Phi(n) \equiv \int_0^n \varphi(i)di$ to be the total weight of the n offsprings relative to the first offspring.

Corollary 6. Let Axiom 3a hold for $n \in [0, N]$. Then $G_n(U, V', D, n) = G_0(U, V', D, 0)\varphi(n)$, where $\varphi(n)$ is a positive strictly decreasing function satisfying $\varphi(0) = 1$ for $0 \leq n \leq N$. Moreover, $G_V(U, V', D, n) = \widehat{G}_0(U, \mathbf{V}_0', \mathbf{D}_N)\Phi(n)$.

The last part of Corollary 6 uses Corollary 1. Examples of two possible functional forms for $\varphi(n)$ are exponential and hyperbolic. Exponential child discounting takes the form $\varphi(i) = e^{-\mu i}$, $\mu > 0$, and implies $\Phi(n) = \left(1 - e^{-\mu n}\right)/\mu$. This type of discounting is the natural counterpart of exponential *time* discounting of dynastic models. It has the convenient property that $\Phi(\infty) = 1/\mu < \infty$ which helps to keep the parental utility bounded for any number of potential children.

Hyperbolic child discounting takes the form $\varphi(i) = i^{-\epsilon}$, $0 < \epsilon < 1$, and implies $\Phi(n) = n^{1-\epsilon}/\left(1 - \epsilon\right)$. The restriction $0 < \epsilon$ is required for altruism to be decreasing and $\epsilon < 1$ is required for $G_V(U, V', D, n) > 0$ (see Corollary 6). Hyperbolic discounting is the more standard assumption in the fertility literature following the original BB formulation. Although it is convenient in some cases, as it allows simple aggregation, it may also be problematic. In particular, $\varphi(0) = \infty$ which violates one of the properties in Corollary 6 meaning that altruism is not strictly decreasing. A practical implication of this feature is that hyperbolic discounting forces an interior solution of fertility choices which may not be desirable given that many individuals choose not to have children.

While marginal altruism, $\widehat{G}_i$, is relevant for fertility decisions, average altruism is relevant for bequests decisions. Average altruism is defined as:

$$\beta(n; U, V', D) \equiv G_V(U, V', D, n)/n. \tag{11}$$

Thus, while $G_V(U, V', D, n)$ is the total increase in parental utility due to one more util for each offspring, $\beta(n; U, V', D)$ is the increase in parental utility per-offspring. It is natural to require that $\beta(n; U, V', D)$ weakly decreases with n so that larger families do not exhibit stronger incentives to leave larger bequests per offspring. The following proposition states that average altruism is decreasing under Axiom 3a.

Proposition 2. Let Axiom 3a hold for $n \in [0, N]$. Then $\beta(n; U, V', D)$ decreases with n.

Proof. Equation (11) and Corollary 6 imply that $\beta(n; U, V', D) = \widehat{G}_0(U, \mathbf{V}'_0, \mathbf{D}_N)\Phi(n)/n$. Function $\Phi(n)$ is strictly concave because $\varphi(i)$ is strictly decreasing. Therefore, $\Phi(n)/n$ decreases with n.

This completes the presentation of the basic axioms of altruism. The following examples illustrate the importance of establishing a set of axioms in deriving the appropriate altruistic preferences to study fertility choice problems. Each example violates at least one of the basic axioms of altruism proposed above. Example 2 illustrates a simple extension of the BB preferences that seek to deal with negative utility functions but turn out to violating Axiom 2, the fundamental axiom of altruism.

Example 2 - Violation of Axiom 2. *Alvarez (1999), Barro and Sala-i-Martin (2004), and Jones and Schoonbroodt (2010) consider preferences of form*

$$V = G(U, V', D, n) = U + n^{\theta} V' \tag{12}$$

where $U < 0$, $\theta < 0$ and D is implicitly normalized to 0. In this case V and V' are negative because they represent present values of negative utility flows ($U < 0$), and the sign *of θ guarantees $G_n = \theta n_t^{\theta-1} V' > 0$. These preferences violate the fundamental axiom of altruism (Axiom 2). To see this, notice that according to equation (10), $G_n = \theta n_t^{\theta-1} V' = \widehat{G}_n(U, V'_n, D_{N-n})(V' - D) = \widehat{G}_n(U, V'_n, D_{N-n})V'$. Therefore, $\widehat{G}_n(U, V'_n, D_{N-n}) = \theta n_t^{\theta-1} < 0$ which violates Axiom 2.*

Example 2 illustrates the importance of the axiomatic approach that links the extensive form of the utility function, $\widehat{G}$, to its reduced form, G. The reason why G violates altruism is because it implicitly assumes that children's welfare is detrimental to parental welfare. Further discussion on this issue is provided in Appendix A. The following extension of the BB preferences seeks to allow for negative and positive utility functions but it violates indifference (Corollary 4).

Example 3 - Violation of Corollary 4. *Jones and Schoonbroodt (2009) consider preferences of form:*

$$V = u(c_t) + n_t^{\theta} V' + (N - n_t)^{\theta} D. \tag{13}$$

If preferences are altruistic, all what should be required for offsprings to be goods, i.e. for $\partial V/\partial n > 0$, is that $V' > D$ (Corollary 4). However, this is not the case with (13). In this case $\partial V/\partial n > 0$ requires the more complicated condition $n_t^{\theta-1} V' > (N - n_t)^{\theta-1} D$. In addition, (13) violates the indifference result stated in Corollary 4 according to which the parent must be indifferent between any feasible n when $V' = D$. An alternative formulation of (13) that satisfies altruism is $V = u(c_t) + n_t^{\theta} V' + \left(N^{\theta} - n_t^{\theta}\right) D$.

The next example illustrates the complications that emerge when writing down altruistic preferences for a fertility choice problem in which survival and death are potential states to be considered.

Example 4 - Another violation of Corollary 4. *Birchenall and Soares (2009) consider preferences of the form:*[11]

$$V = p_a \left[u(c_t) + (\phi n_t)^{\theta} V'\right] + (1 - p_a) M$$

where p_a is the adult survival probability, ϕ is the fraction of offsprings that survive, and M is the value of death. To see the complications that emerge in this type of set up, suppose $V' = M$. In this case parents should be indifferent between any ϕ because offsprings receive the same utility regardless of whether they survive or die. This is not the case in the specification above.

Example 4 features values for three possible states: the value of being alive V, the value of death M, and the value of the unborn child D. A specification consistent with the spirit of the axioms proposed in this section should make altruistic parents indifferent between any number of children n if $V' = D$, and indifferent between any child survival probability ϕ if $V' = M$. This example highlights the importance of well microfounded preferences.

[11] We consider a simpler case in which the emotional cost of lossing a child is set to zero ($M_c = 0$) and assume that a constant fraction of offsprings die.

3.3 Stationarity

We now introduce additional axioms in order to guarantee the existence of a unique stationary solution for the map $V = G(U, V', D, n)$. For this purpose, we focus on the symmetric case described by (9). For any feasible U, n and D, a stationary solution of (9) satisfies:

$$V^* = G(U, V^*, D, n). \tag{14}$$

This equation characterizes a stationary solution in which all generations have the same number of offsprings and attain the same lifetime utility. A desirable property of this solution would be that $V^* > D$ so that born children are goods.

At this stage it is convenient, but not essential, to restrict the space of utility values to be in the positive real line. This is largely without loss of generality because utility functions, such as the CRRA or the CARA, that may take negative values, can be transformed into non-negative functions using monotonic transformations (see examples in Section 3.6 and the discussion there).

Axiom 4 - Positive utilities. $(U, V, D) \in R_+$

Given Axiom 4, it is natural to assume the normalization $\widehat{G}(0, 0, ..., 0) = 0$. This normalization pins down the intercept of function G by stating that parental utility only arises from personal utility or descendants utility, but no from other source.

Axiom 5 - Normalization. $\widehat{G}(0, 0, ..., 0) = 0$.

Corollary 7. $G(0, 0, 0, n) = 0$ for all $0 \leq n \leq N$.

The following assumption guarantees the existence and uniqueness of a stationary solution for the mapping (14):

Axiom 6 - Stationarity. $G(U, 0, D, n) > 0$, $G_{VV} \leq 0$ and $lim_{X\to\infty} G(U, X, D, n) < X$ for any $U > 0$, $0 \leq n \leq N$ and $D \geq 0$.

In words, Axiom 6 states that children are not essential and that there is enough discounting. In particular, $G(U, 0, D, n) > 0$ means that if the utility of all born children is zero, the parent's utility is still positive, so that children are not essential. The assumption that $lim_{X\to\infty} G(U, X, D, n) < X$ implies that even as born children's utility goes to infinity, parents discount is enough so that their own utility does not increase as fast. Graphically, $lim_{X\to\infty} G(U, X, D, n) < X$ implies that on a two-dimensional space that maps X into V (see Figure 2), function G eventually crosses the 45-degree line to allow for the stationary solution $V^* = G(U, V^*, D, n)$ to exist. Notice that $G(U, 0, D, n) > 0$ implies that G crosses the 45-degree line from above. Finally, concavity in V, as implied by $G_{VV} \leq 0$, guarantees that the stationary solution is unique.

3.4 Welfare of the unborn

D can be regarded, in general, as an exogenous parameter. However, a case can be made for D to be derived endogenously in any altruistic framework. A related concept in the social choice literature is that of a "neutral life," a level of utility such that a life is worth living if well-being is above neutrality and is not worth living if well-being is below neutrality (Blackorby, Bossert and Donaldson 2005, p. 25); or that of a "neutral level of wellbeing" a level such that "her living at that level is equally as good as her nonexistence" (Broome, 2004, p. 188). Such value is often normalized to zero in the social choice literature. A similar normalization is also implicit in altruistic models in the BB tradition.

Denote D^* the endogenously determined value of D. Consider the following formalization for D^*. If U collects the utility flow of an individual while alive and $\widetilde{U}$ is "a neutral" flow of utility for which there is no enjoyment nor pain, then D^* can be defined as the present value of such utility flow:

$$D^* = G(\widetilde{U}, D^*, D^*, n). \tag{15}$$

The solution for D^* described by (15) could in principle depend on n but Axiom 3 avoids this dependence. Denote $\underline{U}$ the personal utility associated to a path of zero consumption. We interpret $\underline{U}$ as the utility from non-economic goods, i.e., the utility from all goods not explicitly counted as consumption in the budget constraint (e.g., friendship, public goods, etc). In principle $\widetilde{U} \neq \underline{U}$. For example, $\widetilde{U} > \underline{U}$ means that certain minimum private consumption ($c_t > 0$) is required for neutrality. On the other hand, $\widetilde{U} < \underline{U}$ means that there is enjoyment in life beyond costly consumption due to, say, the existence of non-economic goods (friendship, public goods, etc.). The following assumption guarantees that D^* defined by equation (15) exists and is unique.

Axiom 7 - Value of the unborn child. $\frac{\partial G(\widetilde{U},0,0,0)}{\partial X} < 1$, $G(\widetilde{U}, X, X, 0)$ is concave in X, and $lim_{X\to\infty} G(\widetilde{U}, X, X, 0) < X$.

Corollary 8. $D^* = 0$ if $\widetilde{U} = 0$ and $\partial D^*/\partial\widetilde{U} > 0$ under Axioms 6 and 7.

The interpretation of Axiom 7 is similar to Axiom 6. Figure 1 also portrays the determination of D^*. It naturally follows from the Axioms above that $V^* > D^*$ if $U \geq \widetilde{U}$. To see this, notice that $G(\widetilde{U}, 0, 0, n) < G(U, 0, D, n)$ for any feasible U so that the intercept of function $G(\widetilde{U}, X, X, n)$ lies below that of function $G(U, X, D, n)$. It is not immediately clear which of the two functions cuts the 45-degree line first. It turn out the $G(\widetilde{U}, X, X, n)$ does. To see why, consider function $G(\widetilde{U}, X, D, n)$, which is a parallel shift down of function $G(U, X, D, n)$. By definition, function $G(\widetilde{U}, X, D, n)$ crosses the 45-degree line at the stationary solution D^*, i.e., $D^* = G(\widetilde{U}, D^*, D^*, n)$. Thus, it must be the case that $G(\widetilde{U}, X, X, n) = G(\widetilde{U}, X, D, n)$ at the stationary solution D^* as shown in Figure 1. This result is summarized in the following Proposition.

Proposition 3 - $V^* > D^*$. Let Axioms 1, 4, 5, 6 and 7 hold and $U > \widetilde{U}$. Then $V^* > D^*$.

Corollary 9. Let $V' = V^*$ and $D = D^*$. Then offsprings are normal goods if and only if $U > \widetilde{U}$.

These last results are important because, in general, V' is endogenous. To guarantee that offspring are normal goods requires some underlying restrictions. If the axioms required by Proposition 3 hold then offsprings are normal goods around stationary solutions.

3.5 Marginal rates of substitution

We now study the separate role of U and G in determining three marginal rates of substitution (MRS): *(i)* the willingness to substitute consumption across time for the same individual; *(ii)* the willingness to substitute consumption across individuals, the parent and his children; and *(iii)* the willingness to substitute personal welfare for additional offsprings. To define these rates properly, notice that V can be written solely in terms of utility flows, U, and number of children, n, by recursively substituting V' into equation (9) as:

$$V = G(U, G(U', ...), D, n) \tag{16}$$

3.5.1 Intertemporal and intergenerational substitution of consumption

Consider first the marginal rate of substitution between c_v and c_s, $MRS(c_s, c_v)$, where c_v and c_s are parental consumptions at ages v and s respectively. Since only U depends on c_v and c_s, by definition, then the $MRS(c_v, c_s)$ is given by

$$MRS\left(c_v, c_s\right) = \frac{\partial U/\partial c_v}{\partial U/\partial c_s}.$$

Consider next the marginal rate of substitution between c_v and c'_s, $MRS\left(c_v, c'_s\right)$, where c'_s is the age-s consumption of the offspring. Since c_v only affects U while c'_s only affects U', then $MRS\left(c_v, c'_s\right)$ is given by

$$MRS\left(c_v, c'_s\right) = \frac{\partial V/\partial c_v}{\partial V/\partial c'_s} = \frac{G_U(U, ...) \times \partial U/\partial c_v}{G_V(U, ..) \times G_U(U', ..) \times \partial U'/\partial c'_s}. \tag{17}$$

These marginal rates of substitution can be used to compute elasticities of intertemporal and intergenerational substitution. The EIS is defined as

$$EIS\left(c_v, c_s\right) = \frac{d\ln(c_s/c_v)}{d\ln MRS(c_v, c_s)}, \tag{18}$$

while the EGS can be defined as

$$EGS\left(c_v, c'_s\right) = \frac{d\ln(c'_s/c_v)}{d\ln(MRS\left(c_v, c'_s\right))}. \tag{19}$$

The EIS describes the willingness to substitute consumption intertemporally for the same individual, the parent, while the EGS refers to the willingness to substitute consumption across parents and their children. It is immediate from these definitions that the EIS and the EGS are different in general. While the EIS is fully determined by function U, the EGS is determined by U and G. Therefore, our framework allows to disentangle intertemporal parameters from intergenerational parameters which are typically assumed to be identical. It is also useful to define semi-elasticities, which characterize preferences of the CARA type, as follow:

$$SEIS\left(c_v, c_s\right) = \frac{d(c_s - c_v)}{d \ln MRS(c_v, c_s)}, \text{ and } SEGS\left(c_v, c_s'\right) = \frac{d(c_s' - c_v)}{d \ln(MRS\left(c_v, c_s'\right))}.$$

3.5.2 Marginal rate of substitution between offsprings and consumption

The optimal fertility decision requires to equalize a marginal rate of substitution to a marginal rate of transformation. The willingness to substitute personal welfare for an offspring, or marginal rate of substitution between U and n, is the relevant margin. This margin is fully determined by the function G and it is given by

$$MRS_{n,U}(U, V, D, n) = G_n / G_U. \tag{20}$$

3.6 Examples

This section presents examples of functions U and G satisfying Axioms 1 to 7, and next section derives optimal fertility decision for some these examples. The examples assume a continuous number of children in the interval $[0, N]$ and are written in terms of a generic weighting function $\Phi(n) \equiv \int_0^n \varphi(i)di$, but we discuss specific implications of exponential and hyperbolic weighting. Results are also presented for D exogenous and $D = D^*$. For each case we derive the marginal rate of substitution between composite consumption, U, and offsprings, $MRS_{n,U}$, the average rate of altruism, $\beta(n; U, V, D)$, and the elasticities or semi-elasticities of intertemporal and intergenerational substitution.

The first example, Example 5, is the traditional time-separable CRRA preferences including constants $\underline{U} \geq 0$ and $\widetilde{U} \geq 0$. Example 6 generalizes Example 5 to non-separable CRRA preferences. The separable and non-separable cases differ in two important aspects. First, the separable case imposes the restriction $EIS = EGS$, while the non-separable case allows $EIS \neq EGS$. Second, both preferences can accommodate the case $EIS > 1$ but the separable preferences require non-economic goods ($\underline{U} > 0$) and restrict the consumption space, while the non-separable case does not require such restrictions. These differences give rise to different implications for fertility decisions, as shown in Section 4. Overall, Example 5 is likely the most important for future quantitative work as it offers CRRA preferences suitable to study fertility issues without imposing restrictions on the EIS, the EGS or the consumption space.

Example 7 corresponds to non-separable CARA preferences that disentangle semi-elasticities of intertemporal and intergenerational substitution, $SEIS$ and $SEGS$. Example 8 combines CARA preferences for U with CRRA preferences for G, and Example 9 combines CRRA preferences for U with CARA preferences for G. As it will be shown below in Section 4, the case of CARA preferences is interesting because it allows to obtain a negative fertility-income relationship even with zero non-labor income $Y = 0$ and with no utility from non-economic goods $\underline{U} = 0$. This contrasts with the non-separable CRRA case. Proofs of some of the results in the examples are presented in Appendix B.

Example 5 - Separable CRRA (SCRRA). *Let*

$$U = \int_0^T e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt + \underline{U}, \tag{21}$$

and

$$G(U, V', D, n) = U + \alpha\Phi(n)V' + \alpha\left(\Phi(N) - \Phi(n)\right) D, \tag{22}$$

where $0 \leq \alpha\Phi(N) < 1$, $\sigma > 0$, $\sigma \neq 1$, *and* $c_t \geq \underline{c}(\underline{U}) \equiv \left[\frac{1-e^{-\rho T}}{\rho(\sigma-1)\underline{U}}\right]^{\frac{1}{\sigma-1}}$ *if* $\sigma > 1$. *Some properties of SCRRA preferences are:*

$$MRS_{n,U}(U, V', D, n) = \alpha\Phi'(n)\left(V' - D\right)$$
$$\beta(U, V, D, n) = \beta^*(n) = \alpha\Phi(n)/n$$
$$EIS\left(c_v, c_s\right) = EGS\left(c_v, c_s'\right) = 1/\sigma.$$

The stationary values of D^* *and* V^* *are given by:*

$$D^* = \widetilde{U}/\left(1 - \alpha\Phi(N)\right)$$
$$V^* = \frac{1 - \alpha\Phi(N) + \alpha\left(\Phi(N) - \Phi(n)\right)\left(\widetilde{U}/U\right)}{\left(1 - \alpha\Phi(n)\right)\left(1 - \alpha\Phi(N)\right)} U.$$

Example 5 corresponds to a generalization of equation (1) in Section 2, where the value of unborn children D is explicitly included. Notice that the formulation of $G(U, V', D, n)$ in Example 5 satisfies the fundamental axiom of altruism (Axiom 2), and it also satisfies indifference when $V' = D$ (Collorary 4). The restriction on $\alpha\Phi(N)$ is required to guarantee the existence of a stationary solution. Parameter α determines the degree of altruism towards all descendants. For the exponential discounting case, $\alpha\Phi(N) \leq \alpha/\mu$ so that a sufficient condition for stationarity is $\alpha < \mu$. For the hyperbolic discounting case, $\Phi(\infty) = \infty$ so that the restriction is only satisfied if N is restricted. The second restriction, $c_t \geq \underline{c}(\underline{U})$, is needed to guarantee positive utility when $\sigma > 1$. This condition is not satisfied unless $\underline{U} > 0$, and the larger the $\underline{U}$ the less stringent the restriction is. Notice also that $\underline{U}$ is not a minimum but a maximum utility flow when $\sigma > 1$. The minimum utility flow is 0 and it is obtained when $c_t = \underline{c}(\underline{U})$.

A notable property of SCRRA preferences in Example 5 is that the EIS is the same as the EGS. This is the standard specification that has been used in dynamic altruistic models of fertility. Our next example, Example 6, deviates from this standard by decoupling the EGS from the EIS. It is a generalization of equation (7) in Section 2, where the value of unborn children D is explicitly included.

Example 6 - Non-separable CRRA (NCRRA). *Let*

$$U = \left(\int_0^T e^{-\rho t} c_t^{1-\sigma} dt\right)^{\frac{1}{1-\sigma}} + \underline{U}, \tag{23}$$

and

$$G(U, V', D, n) = \left\{U^{1-\eta} + \alpha\left[\Phi(n)V'^{1-\eta} + (\Phi(N) - \Phi(n))\, D^{1-\eta}\right]\right\}^{\frac{1}{1-\eta}} \tag{24}$$

where $\sigma \geq 0$, $\sigma \neq 1$, $0 \leq \alpha\Phi(N) < 1$, *and* $\eta \geq 0$ *if* $D > 0$ *or* $1 > \eta \geq 0$ *if* $D = 0$. *Some properties of NCRAA preferences are:*

$$\begin{aligned} MRS_{n,U}(U, V', D, n) &= \alpha\Phi'(n)U[\left(V'/U\right)^{1-\eta} - (D/U)^{1-\eta}]/(1-\eta) \\ \beta(U, V', D, n) &= \alpha\Phi(n)\left(V/V'\right)^{\eta}/n \\ EIS\,(c_v, c_s) &= 1/\sigma \end{aligned}$$

and $EIGS\,(c_v, c_s') = 1/\eta$ *if* $\underline{U} = 0$ *and approximately equal to* $1/\eta$ *if* $\underline{U}/U$ *is close to zero. The stationary values of* D^* *and* V^* *are given by:*

$$\begin{aligned} D^* &= \widetilde{U}/\left[1 - \alpha\Phi(N)\right]^{\frac{1}{1-\eta}} \\ V^* &= U\left[\frac{(1 - \alpha\Phi(N)) + \alpha\left(\Phi(N) - \Phi(n)\right)(\widetilde{U}/U)^{1-\eta}}{(1 - \alpha\Phi(n))\,(1 - \alpha\Phi(N))}\right]^{\frac{1}{1-\eta}}. \end{aligned}$$

Several comments regarding Example 6 are in order. First, U is a CES function with elasticity $1/\sigma$ while G is CES function with elasticity $1/\eta$. The restrictions $\sigma \geq 0$ and $\eta \geq 0$ are required for concavity. Second, Example 5 can be obtained as a special case of Example 6 when $\sigma = \eta$ and $\underline{U} = 0$. In this case $EIS = EGS$, which is restrictive to the extent that individuals willingness to substitute consumption intertemporally and intergenerationally may differ. Such distinction may be relevant not only for quantitative work but also for qualitative reasons. Third, contrary to the separable case, U is positive for any value of $\sigma \geq 0$ and any $\underline{U} \geq 0$, a key advantage of the non-separable CRRA. In other words, it is possible to set $\underline{U} = 0$ and still have a well-defined problem for any σ. Fourth, restriction $1 > \eta \geq 0$ if $D = 0$ is required to satisfy Axiom 1. The reason is that descendants become essential goods when $\eta > 1$, and, as a result, $V = 0$ when $D = 0$ regardless of the values of U and V'. This implies that when preferences are non-separable, normalizing $D = 0$ is not without loss of generality, as it implies restricting the EGS to be larger than one. Accommodating $EGS < 1$ requires $D > 0$.

We now turn to analyze examples in the class of CARA utility functions. These examples illustrate the flexibility of the framework we propose to accommodate a number of functional forms that satisfy the axioms of altruism.

Example 7 - Non-separable CARA (NCARA). *Let*

$$U = -\frac{1}{\gamma_U} \ln \left(\frac{\rho}{1 - e^{-\rho T}} \int_0^T e^{-\rho t} e^{-\gamma_u c_t} dt \right) + \underline{U} \tag{25}$$

and

$$G(U, V', D, n) = -\frac{1}{\gamma_G} \ln \left\{ \frac{e^{-\gamma_G U} + \alpha \left[\Phi(n) e^{-\gamma_G V'} + (\Phi(N) - \Phi(n))\, e^{-\gamma_G D} \right]}{1 + \alpha \Phi(N)} \right\}, \tag{26}$$

where $\gamma_U \geq 0$ and $\gamma_G \geq 0$. *Some properties of NCARA preferences are:*

$$\begin{aligned} MRS_{n,U}(U, V', D, n) &= \alpha \Phi'(n) e^{\gamma_G U} \left(e^{-\gamma_G D} - e^{-\gamma_G V'} \right) / \gamma_G \\ \beta(U, V', D, n) &= \frac{1}{n} \frac{\alpha \Phi(n) e^{-\gamma_G (V' - V)}}{1 + \alpha \Phi(N)} \\ SEIS\,(c_v, c_s) &= 1/\gamma_U \\ SEGS\,(c_v, c_s) &= 1/\gamma_G. \end{aligned}$$

The stationary values of D^* and V^* are given by:

$$\begin{aligned} D^* &= \widetilde{U} \\ e^{-\gamma_G V^*} &= \frac{e^{-\gamma_G U} + (\alpha\,(\Phi(N) - \Phi(n)))\, e^{-\gamma_G \widetilde{U}}}{1 + \alpha\,(\Phi(N) - \Phi(n))}. \end{aligned}$$

Regarding Example 7, the restrictions on γ_U and γ_G are required for strict concavity. While Example 7 uses non-separable CARA representations for both U and G, the following to examples combine CARA and CRRA representations for U and G.

Example 8 - NCARA-NCRRA. *Let U be given by (25) and G by (24) where $\gamma_U \geq 0$ and $\eta \geq 0$ if $D > 0$ or $1 > \eta \geq 0$ if $D = 0$. This CARA-CRRA specification satisfies $SEIS\,(c_v, c_s) = 1/\gamma_U$ while other features such as β, $MRS_{n,U}$, V, and D, are those of the NCRRA specification. Finally, $EGS \simeq 1/\eta$ for large consumptions.*

Example 8 carries interesting implications for fertility choices. As we show below, with a CARA representation for U, some of the features needed to generate a negative fertility-income relationship under Example 6, such as positive non-labor income ($Y > 0$) or positive zero-consumption utility ($\underline{U} > 0$) are not necessary under Example 8.

Example 9 - NCRRA-NCARA. *Let U be given by (23) and G be given by (26) where $\gamma_G \geq 0$ and $\sigma \geq 0$. This CRRA-CARA specification satisfies $EIS(c_v, c_s) = 1/\sigma$ while other features such β, $MRS_{n,U}$, V, and D, as are those of the GCARA specification. Finally, $SEGS(c_v, c_s) \simeq 1/\gamma_G$ for large c_v and c_s.*

A variety of other examples can be constructed using monotonic transformations of the utility functions.[12] For instance, (23) is a monotonic transformation of (21) or (25) is a monotonic transformation of $-\frac{1}{\gamma_u}\int_0^T e^{-\rho t} e^{-\gamma_u c_t} dt$, the standard time-separable CARA. However, arbitrary monotonic transformations do not, in general, satisfy some of the basic axioms of altruism, particularly the axiom of identity (Axiom 1).[13]

4 Optimal fertility with non-separability

In this section we revisit the fertility choice problem of Section 2, but we now consider non-separable rather than separable preferences. Recall the problem represents a parent with life span T choosing a life cycle consumption profile $C = [c_0, c_T]$, the number of children $0 \leq n \leq N$, and bequests b' in order to maximize lifetime utility subject to a present value budget constraint and a non-negative bequest constraint. The parent is assumed to have all children at age F. To focus on the fertility decision, we assume that the interest rate is sufficiently low to induce the bequest constraint to bind along the steady state. The resulting model resembles the Samuelson-Diamond OLG economy because bequests are absent, but it differs in that fertility and dynasty size are endogenous. The key issue we study in this section is the ability of altruistic models with non-separable preferences to generate a negative fertility-income relationship.

The parental problem is described by the Bellman equation:

$$V(b) = \max_{0 \leq n \leq N,\ b' \geq 0,\ C=[c_0, c_T]} G(U(C), V(b'), D, n) \tag{27}$$

subject to (2) and (3), where function G satisfies the basic axioms of altruism from Section 3.

[12] Monotonic transfomations preverse ordinal rankings. Standard results in macro typically depend only on the ordinal properties of utility functions and therefore are preserved under monotonic transformations. Results that rely on cardinal properties of utility functions may or may not be preserved.

[13] For instance, the following CARA example fails to satisfy Axiom 1, but satisfies all other axioms. Let

$$U = \frac{1}{\gamma_U}\left(1 - e^{-\rho T} - \rho \int_0^T e^{-\rho t} e^{-\gamma_U c_t} dt\right) + \underline{U}$$

and

$$\begin{aligned} G(U, V', D, n) &= \left(\gamma_U \left(\underline{U} - U\right) + 1 - e^{-\rho T}\right)^{-\gamma_G/\gamma_U} - \left(\gamma_U \underline{U} + 1 - e^{-\rho T}\right)^{-\gamma_G/\gamma_U} \\ &\quad + \alpha\left[\Phi(n)V' + \left(\Phi(N) - \Phi(n)\right)D\right], \end{aligned}$$

where $\gamma_U \geq 0$ and $\gamma_G \geq 0$. The restrictions on γ_U and γ_G are required for strict concavity. Notice in particular that, as required, U of a path of zero consumption is $\underline{U}$ and $G(0,0,0,n) = 0$. Furthermore, $SEIS(c_v, c_s) = 1/\gamma_U$, $SEGS(c_v, c_s) = 1/\gamma_G$. However, $G(U, V', D, 0)$ for $N = 0$ is different from U.

The following assumption bounds the stationary average degree of altruism, $\beta^*(n)$, and thus guarantees that the bequest constraint is binding in steady state.

Assumption 1. $\beta^*(0)e^{rF} \leq 1$.

To gain some intuition about Assumption 1, suppose that G is such that $\beta^*(0) = e^{-\rho F}$ so that a parent with no offsprings discounts the welfare of her first dn offsprings only by their time of birth using the parent's own time discount rate. In this case Assumption 1 becomes $e^{-(\rho - r)F} \leq 1$ or $r \leq \rho$ which is a standard assumption for bequest constraints to bind. If the interest rate is below the rate of time preference, individuals would like to borrow rather than save or give bequests. The condition $\beta^*(0) = e^{-\rho F}$ is obtained, for example, by setting $\alpha = e^{-\rho F}$ in the CRRA cases (see Examples 5, 6 and 8) or $\alpha/(1 + \alpha\Phi(N)) = e^{-\rho F}$ in the CARA cases (see Examples 7 and 9).

Lemma 1. Let Assumption 1 hold. Then the bequest constraint is binding in the steady state.

Proof. Consider a marginal reallocation of consumption from the parent to his n children. The parent reduces his age-0 consumption c_0 in one unit and, in exchange, increases the age-0 consumption of each of his n children c_0' in the amount e^{rF}/n. This reallocation can be obtained by adjusting bequests. The optimality condition for bequests therefore must satisfy the condition $\partial V/\partial c_0 \geq \left(e^{rF}/n\right) \times \partial V/\partial c_0'$, with equality if $b' > 0$. In the steady state,

$$\frac{\partial V/\partial c_0}{\partial V/\partial c_0'} = \frac{\partial V/\partial U \times \partial U/\partial c_0}{\alpha\Phi(n) \times \partial V/\partial V' \times \partial V'/\partial U' \times \partial U'/\partial c_0'} = \frac{1}{n\beta^*(n)}$$

which implies that the bequest constraint is binding if $1 > \beta^*(n)e^{rF}$. If $\beta^*(n)$ is strictly decreasing in n then a sufficient condition for the bequest constraint to bind for any n is $1 \geq \beta^*(0)e^{rF}$.

Given Assumption 1, we can now focus in the steady state situation $b = b' = 0$. Recall from the discussion in Section 2 that it is convenient to solve the remaining problem in two steps. First, find the optimal consumption path given n; and second, solve for n. In absence of bequests and for given n, the optimal consumption plan solves the subproblem:

$$U^*\left(I\left(n\right)\right) = \max_{C=[c_0,c_T]} U(C) \text{ subject to } I(n) \geq \textstyle\int_0^T e^{-rt} c_t dt. \tag{28}$$

Once $U^*\left(I\left(n\right)\right)$ is solved for, the parental problem can be recasted as one of choosing only the number of offsprings:

$$V = \max_{n\in[0,N]} G(U^*\left(I\left(n\right)\right), V', D, n) \tag{29}$$

where the optimality condition is given by

$$\left(-\frac{\partial I}{\partial n}\right) \cdot \frac{\partial U^*}{\partial I} = MRS_{n,U}\,. \tag{30}$$

We now consider the solution to this problem using specific non-separable functional forms for U and G. In particular, we illustrate the predictions of our fertility framework by analyzing the cases of Example 6 and Example 8.

4.1 NCRRA preferences

Consider first the preferences described in Example 6 with U given by (23) and G given by (24). The following lemma characterizes $U^*(I)$ for this case.

Lemma 2 - U* for U=NCRRA. Let U be given by (23). Then

$$U^*(I) = \Theta_2 I + \underline{U} = \Theta_2 \left(wL(n) + Y - n\chi\right) + \underline{U}$$

where

$$\Theta_2 = \left(\frac{r - (r-\rho)/\sigma}{1 - e^{((r-\rho)/\sigma - r)T}}\right)\left(\frac{1 - e^{((r-\rho)(1-\sigma)-\rho)T}}{\rho - (r-\rho)(1-\sigma)}\right)^{\frac{1}{1-\sigma}} > 0.$$

Lemma 2 implies that in the non-separable CRRA specification the marginal utility of income is a constant Θ_2. Using Lemma 2 and the characterization of the $MRS_{n,U}$ given in Example 6 for (24), equation (30) can be written in steady state as:

$$\left(\chi - wL'(n^*)\right)\Theta_2 = \frac{\alpha\Phi'(n^*)}{1-\alpha\Phi(n^*)} U^* \frac{1 - (\widetilde{U}/U^*)^{1-\eta}}{1-\eta} \tag{31}$$

where the left-hand-side represents the marginal cost of children, while the right-hand-side is the marginal benefit. In the non-separable CRRA case the marginal cost of children clearly increases with wages. This increase is linear if $\chi = 0$ and asymptotically linear otherwise. As for the marginal benefit, it also increases with wages because U^* is a function of wages. Whether the marginal benefit increases linearly or not with wages, even asymptotically, depends critically on η. The following two propositions state the main results regarding fertility choices under NCRRA preferences. Proposition 4 states that if $\eta > 1$ or $EGS < 1$ then the number of children approaches N for w sufficiently large. This result is problematic because the evidence indicates that fertility decreases with income. Proposition 5 characterizes the requirements for a negative fertility-income relationship for any wage w.

Proposition 4. Suppose U is defined by (23), G is defined by (24) and let $w \to \infty$. Then $n^* \to N$ if $\eta > 1$ and $n^* \to \widetilde{n}$ if $0 < \eta < 1$ where $\widetilde{n}$ solves the equation:

$$-L'(\widetilde{n})\Theta_2 = \frac{\alpha\Phi'(\widetilde{n})}{1-\alpha\Phi(\widetilde{n})} L(\widetilde{n}) \frac{1}{1-\eta}. \tag{32}$$

Proof. Equation (31) can be written as

$$\left(\chi/w - L'(n^*)\right)\Theta_2 = \frac{\alpha\Phi'(n^*)}{1-\alpha\Phi(n^*)}\frac{U^*}{w}\frac{1-(\widetilde{U}/U^*)^{1-\eta}}{1-\eta}.$$

Let $w \to \infty$. Then $\chi/w \to 0$, $U^*/w \to L(n)$ and $\widetilde{U}/U^* \to 0$. For the case $0 < \eta < 1$ the previous equation becomes (32) for large w. For the case $\eta > 1$ the right-hand-side of the previous expression, which is the marginal benefit of children divided by w, diverges to $+\infty$ leading to maximum number of children.

The intuition for the results in Proposition 4 is interesting. Referring back to equation (30), under NCRRA preferences the marginal utility of income is constant and the marginal costs of children increases with wages. The reason why $n^* \to N$ if $\eta > 1$ occurs because the marginal benefit of children increases too fast with wages. In other words, when $\eta > 1$ the utility gain of the born child, $U^* - \widetilde{U}$, becomes too valuable for the parent when $w \to \infty$. Recall that if $\eta > 1$ then the EGS is low ($EGS < 1$). In this case, if wages go to infinity, parents drive their consumption to infinite as part of the optimal plan. Moreover, the low elasticity of substitution between children's and parental consumption induces the parent to provide an increasing consumption to their children. More importantly, providing consumption to a new child becomes increasingly more valuable than rising consumption of existing family members. For this reason, parents facing a very large wage would like to have as many children as possible if $EGS < 1$.

Next, Proposition 5 characterizes the requirements for $\partial n^*/\partial w$ to be negative for any w. As the proposition indicates, the negative fertility-income relationship can be obtained when $0 < \eta < 1$ and $\Theta_2 Y + \underline{U} > 0$. It could also be obtained for $\eta > 1$ but when $\eta < 1 + \frac{\Theta_2 Y + \underline{U}}{\Theta_2 w L}$. This upper bound decreases with w and becomes 1 as $w \to \infty$.

Proposition 5. Suppose U is defined by (23) and G is defined by (24). Then $\frac{\partial n^*}{\partial w} < 0$ if $\chi = 0$, $\widetilde{U} \simeq 0$ and $\frac{\Theta_2 Y + \underline{U}}{\Theta_2 w L} > \eta - 1$.

Proof. Equation (31) can be written as

$$\ln\left(\chi - wL'(n^*)\right) + \ln\Theta_2 = \ln\alpha + \ln\Phi'(n^*) - \ln\left(1-\alpha\Phi(n^*)\right) + \eta\ln U^* + \ln\frac{U^{*1-\eta} - \widetilde{U}^{1-\eta}}{1-\eta}.$$

Differentiating around the steady state yields:

$$\begin{aligned}\frac{-L'dw - wL''dn^*}{\chi - wL'} &= \left(\frac{\Phi''}{\Phi'} + \frac{\alpha\Phi'}{1-\alpha\Phi}\right)dn^* + \frac{\eta}{U^*}\frac{\partial U^*}{\partial I}dI + \frac{1-\eta}{U^{*1-\eta} - \widetilde{U}^{1-\eta}}U^{*-\eta}\frac{\partial U^*}{\partial I}dI \\ &= \left(\frac{\Phi''}{\Phi'} + \frac{\alpha\Phi'}{1-\alpha\Phi}\right)dn^* + \Psi\frac{\Theta_2}{U^*}\left(\left(wL' - \chi\right)dn^* + Ldw\right)\end{aligned}$$

where

$$\Psi = \eta + \frac{1-\eta}{1-(\widetilde{U}/U^*)^{1-\eta}} = \frac{1-\eta(\widetilde{U}/U^*)^{1-\eta}}{1-(\widetilde{U}/U^*)^{1-\eta}} > 0.$$

Collecting terms:

$$-\left[\frac{wL'}{\chi - wL'} + \Psi_2\Theta_2\frac{wL}{U^*}\right] dw/w = \left[\frac{\Phi''}{\Phi'} + \frac{\alpha\Phi'}{1-\alpha\Phi} + \frac{wL''}{\chi - wL'} + \Psi_2\Theta_2\frac{wL' - \chi}{U^*}\right] dn^*.$$

Next, (31) can be written as $\frac{\alpha\Phi'}{1-\alpha\Phi} = \frac{\Theta_2(\chi - wL')}{U^*}\left[\Psi - \eta\right]$. Replacing this result into the previous equation and simplifying:

$$-\left[\frac{L'}{\chi - wL'} + \Psi\Theta_2\frac{L}{U^*}\right] dw = \left[\frac{\Phi''}{\Phi'} - \eta\Theta_2\frac{\chi - wL'}{U^*} + \frac{wL''}{\chi - wL'}\right] dn^*$$

or

$$\frac{dn^*}{dw} = \frac{\frac{L'}{\chi - wL'} + \Psi\Theta_2\frac{L}{U^*}}{-\frac{\Phi''}{\Phi'} + \eta\frac{\Theta_2(\chi - wL')}{U^*} - \frac{wL''}{\chi - wL'}}$$

Provided that $L'' < 0$, or even if $L'' > 0$ but small, the denominator of this expression is always positive. Therefore, $dn^*/dw < 0$ if and only if

$$\frac{wL'}{\chi - wL'} + \left(\eta + \frac{1}{\frac{1}{1-\eta} - \frac{1}{1-\eta}(\widetilde{U}/U^*)^{1-\eta}}\right)\Theta_2\frac{wL}{U^*} < 0$$

where the first component is negative number between -1 and 0, while the second component is positive for any $\eta > 0$ given that $\widetilde{U} < U^*$. The condition has best chances to be satisfied when $\chi = \widetilde{U} = 0$ because the negative component is as large as possible (-1) and the positive component is as small as possible. If $\chi = \widetilde{U} = 0$ and $0 < \eta < 1$, the previous condition becomes $1 > \Theta_2\frac{wL}{U^*} = \frac{\Theta_2 wL}{\Theta_2(wL(n^*)+Y)+\underline{U}}$ or $\Theta_2 Y + \underline{U} > 0$. If $\chi = \widetilde{U} = 0$ and $1 < \eta$, the expression above becomes $1 > \eta\Theta_2\frac{wL}{U^*} = \eta\frac{\Theta_2 wL}{\Theta_2(wL(n^*)+Y)+\underline{U}} = \eta\frac{1}{1+(\Theta_2 Y+\underline{U})/\Theta_2 wL}$ or $\frac{\Theta_2 Y+\underline{U}}{\Theta_2 wL} > \eta - 1$.

Proposition 5 is one of the main results of this paper. It states that regardless of the EIS, it is possible to obtain a negative fertility-income relationship in a dynamic model of altruism with binding non-negative bequest constraints, as long as either $Y > 0$ or $\underline{U} > 0$ and $EGS > 1$. Recall from the discussion in Section 2 that in the case of separable utility, and provided that either $Y > 0$ or $\underline{U} > 0$, a negative fertility-income relationship could only be obtained when $EIS > 1$. The contribution here is that we have obtained a non-separable specification for which $EIS < 1$ can be assumed, as in most quantitative macro, but a negative fertility-income relationship still holds. In this case, restrictions are placed on the EGS. This elasticity is a new concept and has never been estimated. What our analysis suggests is that $\eta < 1$ would be consistent with fertility and income data. We leave the estimation of η for future work.

4.2 NCARA-NCRRA preferences

The main result in Proposition 5 requires either $Y > 0$ or $\underline{U} > 0$. But this may not be a necessary condition in the context of other functional forms for U and G, as we now turn to analyze. Consider next the preferences in Example 8, which combine a CARA specification for U with a CRRA specification for G. The interesting feature of this example is that, as we show below, it is possible to have a negative fertility-income relationship, even if $Y = \underline{U} = 0$. Consider the case in which U is given by (25) and G is given by (24). The following lemma characterizes U^* for this case.

Lemma 3 - U* for U=NCARA. Let U be given by (25), $r < \rho$ and $I \geq \frac{1}{r^2}\frac{\rho-r}{\gamma}\left(rT + e^{-rT} - 1\right)$. Then,

$$U^*(I) = \Theta_3 I + A + \underline{U}$$

where $\Theta_3 = \frac{r}{1-e^{-rT}}$ and

$$A \equiv \frac{1}{\gamma}\left[\left(\frac{\rho - r}{r}\frac{1 - e^{-rT}(1 + rT)}{1 - e^{-rT}}\right) - \ln\left(\frac{\rho}{r}\frac{1 - e^{-rT}}{1 - e^{-\rho T}}\right)\right] > 0.$$

The resulting indirect utility $U^*(I)$ is very similar to the one in Lemma 2, except for the presence of a new constant $A > 0$ and a restriction for I to be above a certain level. This restriction is required to avoid zero consumption in some periods which may occur because CARA preferences do not satisfy Inada conditions. Given the similarity with the previous section, the following two propositions follow.

Proposition 6. Suppose U is defined by (25), G is defined by (24) and let $w \to \infty$. Then $n^* \to N$ if $\eta > 1$ and $n^* \to \widetilde{n}$ if $0 < \eta < 1$ where $\widetilde{n}$ solves the equation:

$$-L'(\widetilde{n})\Theta_3 = \frac{\alpha\Phi'(\widetilde{n})}{1 - \alpha\Phi(\widetilde{n})}L(\widetilde{n})\frac{1}{1 - \eta}.$$

Proposition 7. Suppose U is defined by (25) and G is defined by (24). Then $\frac{\partial n^*}{\partial w} < 0$ if $\chi = 0$, $\widetilde{U} \simeq 0$ and $\frac{\Theta_3 Y + A + \underline{U}}{\Theta_3 w L} > \eta - 1$.

The novel aspect of Proposition 7 is that it states that $\partial n^*/\partial w$ can be negative even if $Y = \underline{U} = 0$. The fact that $r < \rho$ and the form of consumption smoothing by individuals with CARA preferences induces an effect analogous to the presence of non-labor income or non-economic goods. Proposition 7 illustrates the flexibility of our proposed framework in generating a negative fertility-income relationship in a fully dynamic altruistic model.

5 Demand for longevity

So far we have used the general class of altruistic preferences we are proposing to study fertility choices, but the length of life T has been taken as exogenously given. In this section we address the question of whether or not the preferences we propose preserve the prediction that longevity is positively related to income. As Hall and Jones (2007) show in a standard time-separable framework, when $EIS < 1$ there is a positive longevity-income relationship as in the data. But it is not clear a priori whether this still holds under non-separable preferences of the sort we propose, when $EGS > 1$ and $EIS < 1$. For this purpose we now consider a problem in which an altruistic parent with NCRRA preferences chooses the length of life T.

Consider the following parental problem:

$$V(b) = \max_{C=[c_0,c_T],T,n,b'} \left\{ U(C,T)^{1-\eta} + \alpha \left[\Phi(n) V(b')^{1-\eta} + \left(\Phi(N) - \Phi(n) \right) D^{1-\eta} \right] \right\}^{\frac{1}{1-\eta}}$$

subject to:

$$\begin{aligned} b + wL(T,n) + Y(T) &\geq \textstyle\int_0^T e^{-rt} c_t dt + ne^{-rF} b' + \chi(n,T) \\ b' &\geq 0 \end{aligned}$$

where $U(C,T)$ is the personal utility derived from consumption C and longevity T, and as before, n is number of children, b is bequest, w is wage. $L(T,n)$ is the lifetime labor income which depends negatively on n and also on T. The latter assumption can be justified on the grounds that living longer has time costs such as exercising, cooking healthy meals at home, etc. $Y(T)$ is the present value of non-labor income at time t, a function satisfying $Y'(T) \geq 0$. Last, $\chi(T,n)$ is the present value of the non-labor cost of children and longevity, a function satisfying $\chi_T(n,T) > 0$, $\chi_{TT}(n,T) > 0$ and $\lim_{T\to\infty} \chi(n,T) = \infty$.

Assume further that $U(C,T)$ takes the (non-separable) CRRA form:

$$U(C,T) = \left(\int_0^T e^{-\rho t} c_t^{1-\sigma} dt + \int_T^\infty e^{-\rho t} d^{1-\sigma} dt \right)^{\frac{1}{1-\sigma}}$$

or

$$U(C,T) = \left(\int_0^T e^{-\rho t} c_t^{1-\sigma} dt + d^{1-\sigma} e^{-\rho T}/\rho \right)^{\frac{1}{1-\sigma}} \tag{33}$$

with $\sigma \geq 0$ and $\sigma \neq 1$, and where d is the consumption flow associated to be dead. Notice that $1/\sigma$ is the EIS and regardless of whether $\sigma \gtreqless 1$, $U_T(C,T) \geq 0$ if and only if $c_T \geq d$, a result that is intuitively clear. Also, if $d = 0$ then the restriction $\sigma < 1$ is required to avoid $U(C,T) = 0$, which resembles the restriction on η already discussed for the case of fertility decisions. There is also a similarity with the separable CRRA case analyzed by Hall and Jones (2007) who need to add a constant when $\sigma > 1$ in order to guarantee positive utility. However, in our non-separable context adding the constant $d > 0$ for $\sigma > 1$ is not required to guarantee positive utility, because utility is non-negative even if $d = 0$. In our case $d > 0$ is added to guarantee that utility strictly increases with C and T.

In order to simplify the problem above without adding major costs, we assume $\rho = r > 0$. We also assume that the bequest constraint is binding due to a small degree of altruism. In the absence of bequests and for given n, the solution for C and T can be obtained from the solution to the following subproblem:

$$\widetilde{U}(n) = \max_{C=[c_0,c_T],T} U(C,T) + \text{ subject to } W(T,n) \geq \textstyle\int_0^T e^{-rt} c_t dt$$

with

$$W(T,n) \equiv wL(n,T) + Y(T) - \chi(n,T) \tag{34}$$

where by assumption $W(T,n)$ decreases with T.[14] The number of children can then be solved from the problem

$$V = \max_n \left\{ \widetilde{U}(n)^{1-\eta} + \alpha \left[\Phi(n) V'^{1-\eta} + (\Phi(N) - \Phi(n)) D^{1-\eta} \right] \right\}^{\frac{1}{1-\eta}}.$$

Let λ be the Lagrange multiplier on the budget constraint (34). The optimality conditions for c_t and T are (assuming interior solutions):

$$\frac{\partial U(C,T)}{\partial c_t} = \lambda e^{-rt} \text{ for } t = [0,T] \tag{35}$$

$$\frac{\partial U(C,T)}{\partial T} + \lambda \left[\frac{\partial W(T,n)}{\partial T} - e^{-rT} c_T \right] = 0 \tag{36}$$

From (35) one obtains the standard solution:

$$c_t = c_0 e^{(r-\rho)t/\sigma} = c_0 \tag{37}$$

so consumption is constant given the assumption $r = \rho$. Moreover,

$$\textstyle\int_0^T e^{-rt} c_t dt = c_0 \int_0^T e^{-rt} dt = \left[1 - e^{-rT}\right] c_0.$$

Substituting this result into the budget constraint and solving for c_0 results in:

$$c_0 = c_t = c(T) = \frac{r}{1 - e^{-rT}} W(T,n). \tag{38}$$

This equation characterizes the optimal consumption choices as a function of T. Consumption

[14] An alternative assumption is that $W(T,n)$ first increases with T because living longer is initially inexpensive, but eventually decreases with T because it becomes increasingly costly. The empirical support for the relationship between $W(T,n)$ and T is beyond the scope of this paper and is left for future research.

depends on T in two ways. The first term, $r/(1-e^{-rT})$, decreases with T meaning a longer life span reduces the consumption that can be obtained in each period. In the limit, this terms becomes r. The second term $W(T,n)$ decreases with T.

We now characterize the optimal T. From (35) one obtain $\partial U(C,T)/\partial c_0 = \lambda$. Therefore, condition (36) can be written as:

$$MRS\,(c_0,T) = e^{-rT}c_T - \frac{\partial W(T,n)/\partial T}{W(T,n)}W(T,n) \tag{39}$$

with

$$MRS\,(c_0,T) \equiv \frac{\partial U(C,T)/\partial T}{\partial U(C,T)/\partial c_0} = \frac{\partial c_0}{\partial T}$$

where the left hand side of (39) is the marginal benefit of an additional year of life while the right hand side is the marginal cost, both measured in time-0 consumption units. The marginal cost includes the cost of financing one more year of consumption plus the reduction in wealth.

For the specific utility function in (33), one has that:

$$MRS\,(c_0,T) \equiv \frac{e^{-\rho T}\left(c_T^{1-\sigma} - d^{1-\sigma}\right)}{(1-\sigma)\,c_0^{-\sigma}} = e^{-\rho T}c_T\left(\frac{c_T}{c_0}\right)^{-\sigma}\frac{1-(d/c_T)^{1-\sigma}}{1-\sigma} = e^{-rT}c_T\frac{1-(d/c_T)^{1-\sigma}}{1-\sigma} \tag{40}$$

where the last equality follows from the result that $c_T = c_0$ and $r = \rho$. Substituting (40) into (39), and then substituting (38) and (34) yields

$$\frac{(c_T/d)^{\sigma-1}-1}{\sigma-1} = 1 - \frac{e^{rT}-1}{r}\frac{wL_T(n,T) + Y'(T) - \chi_T(n,T)}{wL(n,T) + Y(T) - \chi(n,T)}. \tag{41}$$

The following proposition examines the relationship between income (wages) and longevity T. It states that if $\sigma > 1$, as income increases individuals choose a higher T.

Proposition 8. $\lim_{w\to\infty} T^* = \infty$ if $\sigma > 1$ while $\lim_{w\to\infty} T^* = \overline{T}$ if $1 > \sigma > 0$ where $\overline{T}$ solves

$$\frac{1}{1-\sigma} = 1 - \frac{e^{rT}-1}{r}\frac{L_T(n,T)}{L(n,T)}$$

Proof. The RHS of equation (41) can be written as:

$$RHS(w) = 1 - \frac{e^{rT}-1}{r}\frac{L_T(n,T) + Y'(T)/w - \chi_T(n,T)/w}{L(n,T) + Y(T)/w - \chi(n,T)/w}$$

so that

$$\lim_{w\to\infty} RHS(w) = 1 - \frac{e^{rT}-1}{r}\frac{L_T(n,T)}{L(n,T)}.$$

On the other hand, since $c_T = \frac{r}{1-e^{-rT}} W(T,n)$ then

$$\lim_{w\to\infty} LHS(w) = \begin{cases} \infty \text{ if } \sigma > 1 \\ \frac{1}{1-\sigma} \text{ if } 1 > \sigma > 0 \end{cases}$$

Therefore, longevity goes to infinite if $\sigma > 1$ because the marginal benefit goes to infinite while the marginal cost remains bounded. Moreover, longevity goes to $\overline{T}$ if $1 > \sigma > 0$.

The following corollary examines the behavior of health expenditures as income grows. It confirms that our non-separable altruistic preferences preserve the prediction in Hall and Jones (2007) that if $\sigma > 1$ (or $EIS < 1$), health expenditures increase with income.

Corollary 10. $\lim_{w\to\infty} \chi(n,T) = \infty$ if $\sigma > 1$ while $\lim_{w\to\infty} \chi(n,T) = \chi(n,\overline{T})$ and $\lim_{w\to\infty} \chi(n,T)/W(n,T) = \lim_{w\to\infty} \chi(n,\overline{T})/W(n,\overline{T}) = 0$ if $1 > \sigma > 0$.

The last part of the corollary follows because $\chi(n,\overline{T})$ becomes constant as w increases while $W(n,\overline{T})$ keeps increasing with w. This corollary is important for the following reason. It highlights the fact that the standard time-separable model in which $\eta = \sigma$ and $\sigma > 1$ cannot account for both the positive longevity-income relationship and the negative fertility-income relationship because the latter requires $0 < \sigma < 1$. In contrast, disentangling η from σ allows to set $\sigma > 1$ as is standard in quantitative macroeconomics; to model health as a superior good as in Hall and Jones (2007); and to get the negative fertility-income relationship as in the data.

6 Concluding comments

As Jones, Schoonbroodt and Tertilt (2008) conclude in their review paper, it is very challenging for dynamic models of parental altruism to correctly predict the negative fertility-income relationship observed in the data. Our paper contributes to this effort. The distinguishing feature of our framework is that it provides an axiom-based class of dynamic altruistic preferences that can be used to analyze fertility choice problems without imposing restrictions on key parameters such as the EIS. Static models where the time-cost of children is the main determinant of fertility require restrictions on the income elasticity of demand for children to generate a negative fertility-income relationship. Almost all existing dynamic altruistic models of fertility, starting from BB all the way to Manuelli and Seshadri (2009), need to assume $EIS > 1$ in order to obtain a well-defined fertility choice problem. However, most quantitative macroeconomic models assume $EIS < 1$. Our framework resolves these conflicting scenarios by disentangling the EIS from the EGS. This disentangle captures a dimension of intergenerational consumption allocation that had not been singled out before.

The concept of an EGS that differs from the EIS is new. The standard altruistic parent with time-separable utility places a weight on the utility level of his children, but the rate at which parental and children's consumption is substituted is no different that the rate at which parental consumption is substituted across time. Our non-separable dynamic altruistic framework allows to capture an additional dimension of family decision making: the "curvature" that governs the rate at which parents substitute own consumption and children's consumption. We think this additional dimension is of relevance, not only in fertility choice problems, but to study a number of problems in family economics.

We leave for future work the estimation of the EGS as well as some interesting extensions of our framework that are beyond the scope of this paper. Children's schooling decisions have been incorporated in available fertility choice models as part of the so-called "quantity-quality" trade-off, and could also be included in our framework. In our companion paper (Cordoba and Ripoll, 2011) we show that such an extension in the time-separable preferences case is consistent with the quantity-quality trade-off. Similar results should go through in the non-separable framework we study here. In addition, our non-separable dynamic model of parental altruism may be useful to tackle and revisit a number of other interesting issues in macroeconomics, in particular those that involve allocation of resources among parents and children, such as fiscal and distributional issues. From the individual's point of view, the basic unit where many important economic decisions are made is the family. These decisions, such as consumption, saving, number of children, schooling and parental transfers to each of the children usually have long-lasting impact in future economic outcomes of the family members. The use of microfounded dynamic models of parental altruism may provide useful results that are informative for policy makers in influencing the choices made by the family unit.

A Children as longevity

The idea that altruistic parents need to consider the utility of children in the unborn state, D, may seem counterintuitive. It also raises the question of how to identify this parameter (or the utility flow of the unborn, $\widetilde{U}$) in applied work. An alternative way to derive preferences for children by altruistic parents that sheds lights on both issues is to define purely altruistic parents as individuals who regard children as no more or no less than themselves, as an extension of their own life. If so, then fertility decisions are analogous to longevity decisions. Since preferences for longevity are better understood, one can use this equivalence to derive preferences for children.

To formalize this idea, consider an individual, Diana, who lives for up to two days, consuming during the day and sleeping during the night. Diana lives and consumes for sure during the first day. For the second day, Diana has two options, a longevity and fertility option. The longevity option is the following. By investing x resources during the first day, Diana can buy a probability $p(x)$ of awakening, or living, for a second day. With probability $1 - p(x)$ Diana does not awake, or dies. Dying is painless because it is just remaining asleep during the second day. To be more specific, suppose Diana's welfare under this longevity option is described by

$$V^a(c_1, c_2, x) = u(c_1) + p(x)u(c_2) + (1 - p(x))\, u(0), \tag{42}$$

where c_t is day$-t$ consumption and u is a standard concave utility function. The proper description of Diana's welfare includes the term $u(0)$, the utility in the event of not awakening for a second day, or the utility in the death state. Such utility is relevant when deciding x.

The second option, the fertility option, is the following. Diana has no chances of awakening for a second day. Instead, by investing x resources during the first day, Diana can buy a probability

$p(x)$ of creating a new individual, Sophia, Diana's child. If born, Sophia will only live for day 2, will have no children, and will enjoy consumption just as much as Diana does (u is the same for Diana and Sophia). Sophia is in every aspect a replica of Diana in the second day.[15] Let $V^b(c_1, c_2, x)$ be Diana's welfare in the second option where c_1 is Diana's consumption in day 1 and c_2 is Sophia's consumption in day 2.

Whether Diana is altruistic toward Sophia or not depends on how $V^b(c_1, c_2, x)$ compares to $V^a(c_1, c_2, x)$. For example, $V^b(c_1, c_2, x) > V^a(c_1, c_2, x)$ for all (c_1, c_2, x) describes a particularly strong degree of altruism toward Sophia while $V^b(c_1, c_2, x) < V^a(c_1, c_2, x)$ describes a weaker degree of altruism. The borderline case of $V^b(c_1, c_2, x) = V^a(c_1, c_2, x)$ describes a situation in which the parent values her child just as much as she values herself. We call this situation *pure altruism*. According to this definition, a pure altruistic individual will not care whether she or her child is the one who lives during the second day. This definition of altruism immediately implies that (42) also represents Diana's purely altruistic preferences in the fertility problem but with c_2 representing Sophia's consumption instead of Diana's. This result could also be derived from introspection. Suppose Diana's only option is option 2 and her preferences are described by $V^b(c_1, c_2, x)$. To see if $V^b(c_1, c_2, x)$ is purely altruistic or not, Diana can wonder what her welfare would be if she could take Sophia's place and live one more day with probability x. Using this introspection procedure Diana arrives to (42). Pure altruism results if $V^a(c_1, c_2, x) = V^b(c_1, c_2, x)$ for all (c_1, c_2, x).

The equivalence between the longevity and fertility problem for a pure altruistic parent means also that the utility of the unborn child is $u(0)$, that is, the utility of the parent in the death state. This follows because in the formulation above all utility comes from consumption, and being death or unborn both entail zero consumption. While it is well-accepted that the utility in the death state must be explicitly considered in models of endogenous longevity, it is not standard, and it is even controversial, that the utility of children in the unborn state must also be considered. The previous derivation shows that such value arises naturally in altruistic models of fertility. Finally, a typical normalization in longevity models is to set the value of death to zero. The analogy between longevity and fertility implies that in that case the value of the unborn child should also be normalized to zero.

To develop some further implications, suppose that

$$p(x) = \begin{cases} 0 \text{ if } x < \overline{x} \\ 1 \text{ if } x \geq \overline{x} \end{cases}, \tag{43}$$

and

$$c_1 + c_2 + x = W, \tag{44}$$

where W is amount of resources available to the parent. In this formulation $p(x)$ can be interpreted as the number of children, which is either 0 or 1, and $\overline{x}$ is the cost of raising a child. The parent's problem is to maximize (42) subject to (43) and (44). The solution to the problem is:

$$\begin{cases} x = \overline{x},\ p = 1 \text{ and } c_1 = c_2 = (W - \overline{x})/2 \text{ if } 2u((W - \overline{x})/2) \geq u(W) + u(0) \\ x = 0,\ p = 0,\ c_1 = W \text{ and } c_2 = 0 \text{ otherwise.} \end{cases}$$

Thus, the child is born if $\overline{x}$ and $u(0)$ are not "too" high. Moreover, the solution depends on the utility of the child if unborn, $u(0)$, but the sign of $u(.)$ plays no role.

We now discuss the formulation proposed by Alvarez (1999), Barro and Sala-i-Marti (2004), and Jones and Schoonbroodt (2010) in the context of the example above. They specify the welfare of the parent as

$$V^b(c_1, c_2, x) = u(c_1) + p(x)u(c_2). \tag{45}$$

[15] Thus, for example, the fact that Mary would be two days old while Sophia would be one day old plays no role.

In this formulation, the utility of the child if unborn is not explicitly considered. Jones and Schoonbroodt (2009), recognizing this assumption, refer to this specification as one of partial altruism. Both specifications are equivalent when $u(0) = 0$ which also means that $u(c) \geq 0$ is required. Differences arise when when $u(c)$ is negative and therefore $u(0) < 0$. This is the case, for example, when $u(c)$ is of the CARA form or the CRRA form with elasticity below one.

To highlight the consequences of assuming (45) rather than (42) suppose that $u(c) < 0$. In that case, reducing $p(x)$ increases welfare because it reduces the negative effect of an additional period of negative utility flow. If $p(x)$ is increasing in x, as in (43), then $x = 0$ will always be optimal meaning that having no children will be optimal. To avoid this issue Alvarez (1999), Barro and Sala-i-Martin (2004) and Jones and Schoonbroodt (2010) propose assuming that $p(x)$ decreases rather than increases with x.[16] Such assumption implies that $x = 0$ is not necessarily optimal anymore. For example, suppose that

$$p(x) = \begin{cases} 1 \text{ if } x < \overline{x} \\ 0 \text{ if } x \geq \overline{x} \end{cases} . \tag{46}$$

In this case $\overline{x}$ is the cost of child prevention. It is easy to check that the solution is:

$$\begin{cases} x = \overline{x},\ p = 0,\ c_1 = W - \overline{x} \text{ and } c_2 = 0 \text{ if } u(W - \overline{x}) \geq 2u(W/2) \\ x = 0,\ p = 1,\ c_1 = c_2 = W/2 \text{ otherwise.} \end{cases}$$

so the child is born if $\overline{x}$ is high enough.

Although assuming that $p(x)$ decreases with x solves the technical problem of making interior solutions possible, the problem has an unsettling interpretation. Consider first the scenario in which $p(x)$ is a survival probability. In that case, the individual still prefers lower $p(x)$ meaning lower survival probability but dying is costly. In other words, the model becomes a model of "pain" in which the individual would like to end life but dying is costly. Such model may certainly describe very dire situations of depression and/or painful illness but not the typical situation of life as a joy. Consider next the second scenario in which $p(x)$ is the number of children (either 0 or 1). In that case, children inflict a pain to the parent and she would prefer to have no children, but controlling fertility is costly. The interpretation of this fertility choice problem seems at odds with the nature of altruistic models of fertility.

B Proofs

B.1 Example 6

To see that $EGS = 1/\eta$ in the non-separable CRRA case (NCRRA), notice that $\partial U/\partial c_v = e^{-\rho v} (U - \underline{U})^{\sigma} c_v^{-\sigma}, G_U(U, ..) = V^{\eta} U^{-\eta}$ and $G_V(U, ..) = \alpha (1 - e^{-\mu n}) V^{\eta} V'^{-\eta}$. Denote $\widehat{U} = U - \underline{U}$.

[16] Specifically, their formulation is of the type

$$V^b(c_1, c_2, x) = u(c_1) + \beta(n) u(c_2),$$

where n is the number of children. They propose to assume $\beta'(n) < 0$ if $u(c) < 0$. Moreover, in their formulation $x = bn$ where b is the cost of raising a child, a parameter. Therefore, their proposal implies that β decreases with x when $u < 0$.

Therefore,

$$\begin{aligned} MRS\left(c_v,c_s'\right) &= \frac{G_U(U,...)}{G_V(U,..)\times G_U(U',..)}\times\frac{\partial U/\partial c_v}{\partial U'/\partial c_s'}=\frac{V^{\eta}U^{-\eta}}{\alpha\left(1-e^{-\mu n}\right)V^{\eta}V'^{-\eta}\times V'^{\eta}U'^{-\eta}}\times\frac{\partial U/\partial c_v}{\partial U'/\partial c_s'} \\ &= \frac{U^{-\eta}}{\alpha\left(1-e^{-\mu n}\right)U'^{-\eta}}\times\frac{\partial U/\partial c_v}{\partial U'/\partial c_s'}=\frac{U^{-\eta}\widehat{U}^{\sigma}c_v^{-\sigma}e^{-\rho(v-s')}}{\alpha\left(1-e^{-\mu n}\right)U'^{-\eta}\widehat{U}'^{\sigma}c_s'^{-\sigma}}. \end{aligned}$$

Since $\widehat{U}$ is constant returns to scale, it can be written as $\widehat{U}=c_v\widehat{U}_v$ where $\widehat{U}_v$ is homogeneous of degree zero. Therefore,

$$\begin{aligned} MRS\left(c_v,c_s'\right) &= \frac{\widehat{U}_v^{\sigma}e^{-\rho(v-s')}}{\alpha\left(1-e^{-\mu n}\right)\widehat{U}_s'^{\sigma}}\times\left(\frac{U}{U'}\right)^{-\eta}=\frac{\widehat{U}_v^{\sigma}e^{-\rho(v-s')}}{\alpha\left(1-e^{-\mu n}\right)\widehat{U}_s'^{\sigma}}\times\left(\frac{\widehat{U}}{\widehat{U'}}\right)^{-\eta}\left(\frac{1+\underline{U}/\widehat{U}}{1+\underline{U}/\widehat{U'}}\right)^{-\eta} \\ &= \frac{\widehat{U}_v^{\sigma-\eta}e^{-\rho(v-s')}}{\alpha\left(1-e^{-\mu n}\right)\widehat{U'}_s^{\sigma-\eta}}\left(\frac{1+\underline{U}/\widehat{U}}{1+\underline{U}/\widehat{U'}}\right)^{-\eta}\times\left(\frac{c_v}{c_s'}\right)^{-\eta} \end{aligned}$$

so that

$$EGS\left(c_v,c_s'\right)=\frac{d\ln(c_s'/c_v)}{d\ln(MRS\left(c_v,c_s'\right))}\simeq\frac{1}{\eta}\text{ if }\underline{U}/\widehat{U}\approx 0.$$

Specifically, the EGS measures the percentage change in c_s'/c_v due to one percent change in $MRS\left(c_v,c_s'\right)$ holding constant all other consumption ratios.

B.2 Example 7

To see that the semi-elasticity of intertemporal substitution is $SEIS=1/\gamma_U$ in the non-separable CARA case (GCARA), notice that $\partial U/\partial c_v=e^{-\gamma_U(c_v-U+A_U)-\rho v}$ where $A_U=\underline{U}-\frac{1}{\gamma_U}\ln\left[\frac{\rho}{1-e^{-\rho T}}\right]$. Therefore, $MRS\left(c_v,c_s\right)=\frac{\partial U/\partial c_v}{\partial U/\partial c_s}=e^{-\rho(v-s)}e^{-\gamma_U(c_v-c_s)}$ and $SEIS=\frac{d\ln(c_s-c_v)}{d\ln(MRS(c_v,c_s))}=1/\gamma_U$. Thus, the SEIS measures the change in c_s-c_v due to one percent change in $MRS\left(c_v,c_s\right)$. To see that the intergenerational semi-elasticity $SEGS=1/\gamma_G$, notice that $G_U(U,..)=e^{-\gamma_G(U-V+A_G)}$ and $G_V(U,..)=\alpha\left(1-e^{-\mu n}\right)e^{-\gamma_G(V'-V+A_G)}$ where $A_G=\frac{1}{\gamma_G}\ln\left[1+\alpha e^{-\rho F}\left(1-e^{-\mu N}\right)\right]$. Therefore,

$$\begin{aligned} MRS\left(c_v,c_s'\right) &= \frac{G_U(U,...)}{G_V(U,..)\times G_U(U',..)}\times\frac{\partial U/\partial c_v}{\partial U'/\partial c_s'} \\ &= \frac{e^{-\gamma_G(U-V+A_G)}}{\alpha\left(1-e^{-\mu n}\right)e^{-\gamma_G(V'-V+A_G)}e^{-\gamma_G(U'-V'+A_G)+A_G}}\times\frac{e^{-\gamma_U(c_v-U+A_U)-\rho v}}{e_s^{-\gamma_U(c_s'-U'+A_U)-\rho s\prime}} \\ &= \frac{e^{-(\gamma_G-\gamma_U)(U-U')}}{\alpha\left(1-e^{-\mu n}\right)e^{-\gamma_G A_G}}\times e^{-\gamma_U(c_v-c_s')-\rho(v-s')} \end{aligned}$$

On the other hand,

$$\begin{aligned} U-U' &= -\frac{1}{\gamma_U}\ln\left(e^{-\gamma_u c_v}\int_0^T e^{-\rho t}e^{-\gamma_u(c_t-c_v)}dt\right)+\frac{1}{\gamma_U}\ln\left(e^{-\gamma_u c_s'}\int_0^T e^{-\rho t}e^{-\gamma_u(c_t'-c_s')}dt\right) \\ &= c_v-c_s'+M \end{aligned}$$

where

$$M=-\frac{1}{\gamma_U}\ln\left(\int_0^T e^{-\rho t}e^{-\gamma_u(c_t-c_v)}dt\right)+\frac{1}{\gamma_U}\ln\left(\int_0^T e^{-\rho t}e^{-\gamma_u(c_t-c_v)}dt\right).$$

Using this result into the expression for MRS results in

$$MRS\left(c_v, c_s'\right) = \frac{e^{-\gamma_G M}}{\alpha\left(1-e^{-\mu n}\right)e^{-\gamma_G A_G}} e^{-\rho(v-s')} e^{-\gamma_G (c_v - c_s')}$$

so that

$$SEGS = \frac{d\ln(c_s' - c_v)}{d\ln(MRS\left(c_v, c_s'\right))} = 1/\gamma_G$$

where $SEGS$ measures the change in $c_s' - c_v$ due to one percent change in $MRS\left(c_v, c_s'\right)$ holding constant all other consumption differences.

B.3 Example 8

To see that $EGS \approx 1/\eta$ for large consumptions, notice that

$$U = c_v - \frac{1}{\gamma_U}\ln\left(\int_0^T e^{-\rho t} e^{-\gamma_U (c_t - c_v)} dt\right) + A_U = c_v + M_v$$

where

$$M_v = -\frac{1}{\gamma_U}\ln\left(\int_0^T e^{-\rho t} e^{-\gamma_U (c_t - c_v)} dt\right) + \frac{1}{\gamma_U}\ln\left(\frac{1-e^{-\rho T}}{\rho}\right) + \underline{U}.$$

Therefore,

$$\begin{aligned} MRS\left(c_v, c_s'\right) &= \frac{U^{-\eta}}{\alpha\left(1-e^{-\mu n}\right)U'^{-\eta}} \times \frac{\partial U/\partial c_v}{\partial U'/\partial c_s'} = \frac{U^{-\eta}}{\alpha\left(1-e^{-\mu n}\right)U'^{-\eta}} \times \frac{e^{-\gamma_U (c_v - U + A_U) - \rho v}}{e^{-\gamma_U (c_s' - U' + A_U) - \rho s}} \\ &= \frac{U^{-\eta} e^{\gamma_U (U - U')}}{\alpha\left(1-e^{-\mu n}\right)U'^{-\eta}} e^{-\gamma_U (c_v - c_s') - \rho(v-s)} = \frac{U^{-\eta} e^{\gamma_U (c_v - c_s' + M_v - M_s')}}{\alpha\left(1-e^{-\mu n}\right)U'^{-\eta}} e^{-\gamma_U (c_v - c_s') - \rho(v-s)} \\ &= \frac{e^{\gamma_U (M_v - M_s')}\left(U/U'\right)^{-\eta}}{\alpha\left(1-e^{-\mu n}\right)} e^{-\rho(v-s)} = \frac{e^{\gamma_U (M_v - M_s')} e^{-\rho(v-s)}}{\alpha\left(1-e^{-\mu n}\right)}\left(\frac{c_v + M_v}{c_s' + M_s'}\right)^{-\eta}. \end{aligned}$$

where for large consumptions

$$MRS\left(c_v, c_s'\right) \simeq \frac{e^{\gamma_U (M_v - M_s')} e^{-\rho(v-s)}}{\alpha\left(1-e^{-\mu n}\right)}\left(\frac{c_v}{c_s'}\right)^{-\eta}$$

case in which $EGS \simeq 1/\eta$.

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Figure 1. Fertility and Life Expectancy versus Income (2004)

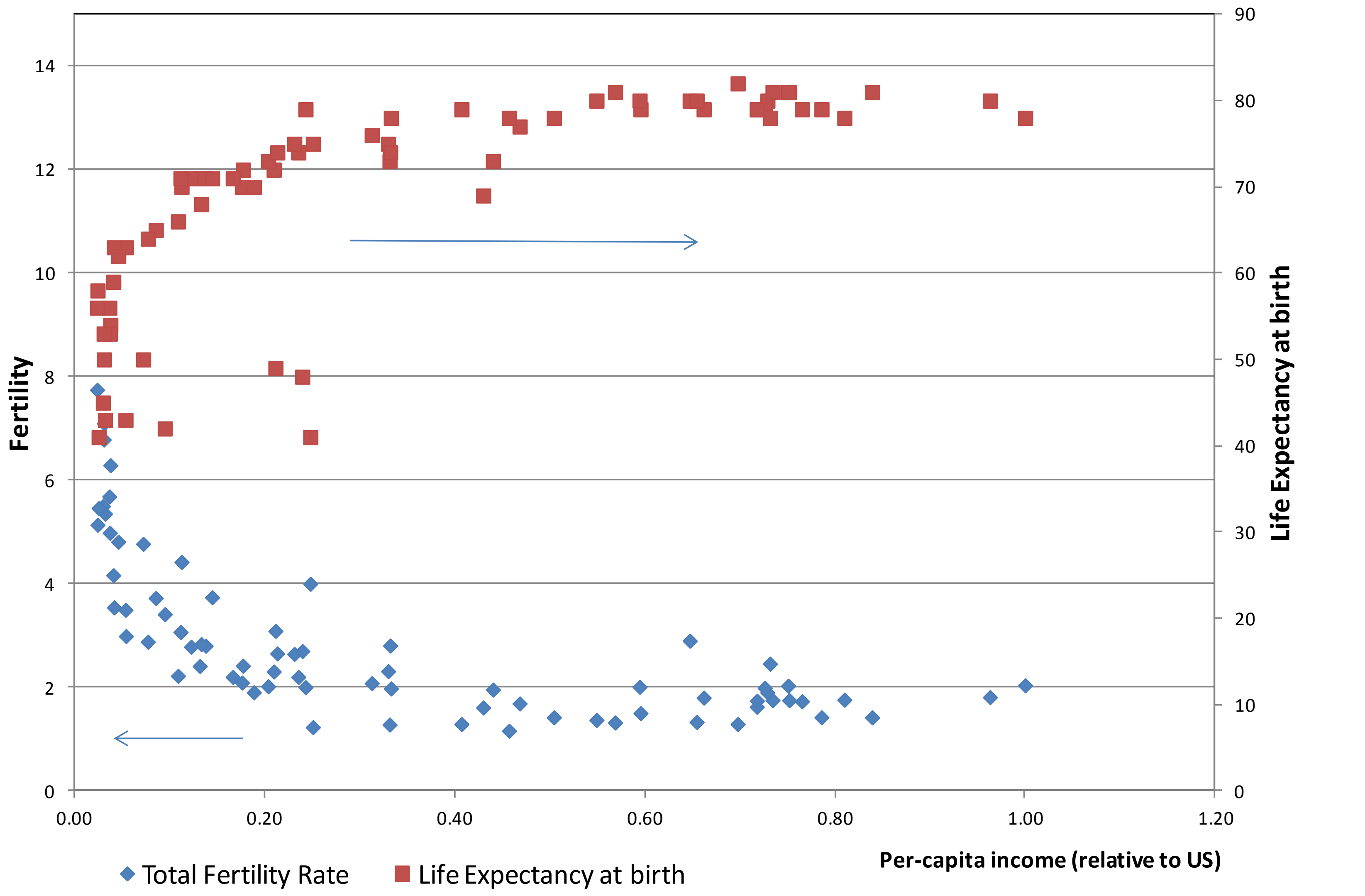

Figure 2. Stationary solutions for ***V*** and ***D***