Chapter 2

Robinson Crusoe

This chapter studies the simplest general equilibrium model one can imagine: an isolated individual -or an isolated family- in an island. A motivation for the story in this chapter is the economic problem faced by Robinson Crusoe, a fictional character in the famous book by Daniel Defoe, who spent 28 years alone in an island. Motivating a model using a simple well-defined environment is a distinguishing feature of modern macroeconomics. It is not unusual that candidates in recruitment seminars in top schools start their presentations with a simple story like the one in this chapter. Crusoe faces the non-trivial economic decision of how much effort to exert in production. In particular, he must allocate his time between a pleasant activity, leisure, and an unpleasant activity, work. Having no social interactions, his actions fully determine all aggregate variables we can think of, and those actions are themselves function of fundamental forces in the island, such as weather and endowments, but also Crusoe's own abilities and preferences.

The simple model in this chapter is extremely valuable for various reasons. First, it allows us to introduce basic notation, concepts, as well as mathematical, statistical, and computational techniques. Second, some insights obtained from this simple framework turn out to be surprisingly robust to a number of seemingly major changes. In particular, the model sheds light on one the most contentious issues in macroeconomics, if not the most: the origins of economic fluctuations. Some theories pose that economic fluctuations primarily originate in the demand side of the economy while others content that it originates in the supply side. The answer to this question is crucial for policymakers because it points to different policy recommendations, if any. Although in the simple model of this chapter any intervention is not optimal, as we show, in the larger context of macro disputes, a case for active government intervention is typically advocated when fluctuations originate on the demand side of the economy.

Third, the chapter also illustrates some of the key challenges in macroeconomics. Even a well defined and simple model can provide very different answers, say about the origins of business cycles and economic growth, depending on how the parameters of the model are identified. The chapter offers three alternative and seemingly reasonable calibrations of the model and illustrates the startling different conclusions.

2.1 The Environment

An individual, Robinson Crusoe, lives alone in an island. He likes fish but dislikes fishing. Fish is non-storable, say because there is no electricity, and the only costly good in the island: other goods like bananas, coconuts, and water are freely available. Crusoe lives for $T+1$ periods, where a period could be a month, a quarter, a year, etc., depending on the specific application of the model. Sometime we consider the case $T=\infty$, meaning that Crusoe lives forever. This assumption is a simple way to study economies that last forever, economies populated by infinitely-lived families or dynasties, rather than individuals.[1]

2.1.1 Preferences

Denote c_t and l_t the consumption of fish and hours of labor, or simply consumption and labor, in period t where $t=0,1,...,T$. Suppose that the total amount of time available for both work and leisure during a period is 1, a normalization that is easy to relax. Thus, $1-l_t$ is the amount of leisure at time t. Denote $\mathbf{c}\equiv\{c_0,c_1,..,c_T\}$ and $\mathbf{l}\equiv\{l_0,l_1,..,l_T\}$ a particular vector -also denoted sequence, path or allocation - of consumption and labor. As a rule, boldface letters will denote vectors rather than scalars. Assume that Crusoe derives welfare - or happiness, utility or pleasure- from consumption and disutility from working and denote $V(\mathbf{c},1-\mathbf{l})$ the lifetime welfare or lifetime utility attached to a particular sequence of consumption and labor.

[1] I explore the equivalence between infinitely-lived agents and dynastic altruism in "Children, Dynastic Altruism, and the Wealth of Nations," *Review of Economic Dynamics* 18:4, October 2015.

In order to obtain explicit predictions that can be compared with actual data, it is necessary to consider specific functional forms for preferences and technologies. The most common specification of preferences postulates that $V(\mathbf{c}, 1-\mathbf{l})$ has the additive separable form:

$$V(\mathbf{c}, 1-\mathbf{l}) = \sum_{t=0}^{T} \beta^t u_t(c_t, 1-l_t). \tag{2.1}$$

In this formulation, lifetime utility V is a discounted sum of utility flows, $\beta^t u_t(c_t, 1-l_t)$, and each utility flow depends only in the consumption and leisure of that period. In this formulation, β^t is the particular weight associated to the utility flow at time t, where β is a number typically between 0 and 1 known as the *discount factor.* Thus, for example, if $\beta = 1$ then $\beta^t = 1$ for all t meaning that utility in all periods is weighted equally. If $\beta = 0.5$, then $\beta^0 = 1$, $\beta^1 = 0.5$, $\beta^2 = 0.25$ and so on, so that later periods weight less in the determination of welfare.

As for the utility flow, $u_t(c_t, 1-l_t)$, it is natural to assume that it is increasing in both arguments. It is also natural to impose some further restrictions on u so that Crusoe prefers interior to corner solutions. For this purpose, it is common to assume that u is strictly concave in both arguments and that it satisfies Inada conditions which basically state that marginal utilities or disutilities at the corners are either zero or infinite. I leave some mathematical details for the Appendix. To focus on the economics of the problem, it is convenient to start with specific functional forms, derive a set of basic, or benchmark, results, and later study whether the results are robust to various generalizations. Assume for the remaining of this section that $u_t(c_t, 1-l_t)$ takes the form:

$$u_t(c_t, 1-l_t) = \lambda_t \ln c_t + \ln(1-l_t), \text{ where } \lambda_t > 0. \tag{2.2}$$

This specification is popular among DSGE practitioners, one that not only satisfies basic requirements of utility functions but also lead to very simple solutions. As discuss below, income and substitution effects exactly offset each other with this formulation. λ_t measures the relative importance of consumption at time t, and may change over time for diverse reasons, such as Crusoe's age. For our purposes, varying λ_t is a simple way to introduce fluctuations on the demand side of the economy. For example, a fall in λ_t may represent a deterioration of "consumer confidence" of the sort that is often heard in the news.

2.1.2 Technologies

In addition to the preferences described above, Crusoe faces technological constraints. Suppose Crusoe catches A_t pounds of fish per unit of time. The variable A_t is called the productivity of labor at time t. This productivity may fluctuate over time due to, say, weather conditions, seasonal fish abundance, Crusoe's health, etc. Productivity can also increase or decrease systematically over time due to learning, experience, or to the discovery of better technologies, say the use of arrows rather than hands, or nets rather than arrows, aging, etc.

2.1.3 Resource constraints

Since fish is perishable and Crusoe has no other source of fish other than his own catch, then the following restriction must hold:

$$A_t l_t \geq c_t \text{ for } t = 0, 1, ..., T. \tag{2.3}$$

This inequality is an example of a *resource constraint.* The resource constraint would be different, for example, if fish could be stored or imported from other islands. These possibilities are studied in later chapters.

2.2 Crusoe's Problem

In order to derive predictions about production, labor, and consumption, further assumptions about Crusoe's behavior are needed. We assume that Crusoe is rational in the following sense: Crusoe's actions are determined by his desire to attain the highest welfare possible. If so, Crusoe's problem can be formalized as:

$$\max_{\{c_t, l_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t \left[\lambda_t \ln c_t + \ln(1 - l_t)\right] \text{ subject to } A_t l_t \geq c_t \text{ for } t = 0, 1, 2, .., T. \tag{2.4}$$

In words, Crusoe chooses the allocation of consumption and labor, $\{c_t, l_t\}_{t=0}^{T}$, that delivers the highest welfare possible subject to resource constraints, one for each period. An *optimal allocation* is one that solves the problem, that is, the maximizer of the problem. We denote optimal allocations by $\{c_t^*, l_t^*\}_{t=0}^T$ or, equivalently, by $[\mathbf{c}^*, \mathbf{l}^*]$.[2]

Problem (2.4) is a simple macroeconomic model: it describes how key macroeconomic variables such a production, $y_t^* \equiv A_t l_t^*$, consumption, labor and social welfare $V(\mathbf{c}^*, \mathbf{l}^*)$ are determined in Crusoe's island in terms of *fundamentals,* to be described next.

2.3 Endogenous versus Exogenous Variables

Crusoe can choose $\mathbf{c}$ and $\mathbf{l}$ but has no control over $\mathbf{A}$, $\boldsymbol{\lambda}$, β or T. As in any model, one has to distinguish between endogenous variables, or unknowns, and exogenous variables, or knows. Endogenous variables are those explained by the model while exogenous variables, or fundamentals, are those taken as given, not affected by endogenous variables. In Crusoe's problem, the endogenous variables are $\mathbf{c}$ and $\mathbf{l}$ while the exogenous variables are $\mathbf{A}$, $\boldsymbol{\lambda}$, β and T. Exogenous variables that are constant over time, such as β and T, are called *parameters*.

On a more conceptual basis, if one denotes all endogenous variables of a model by Y and all exogenous variables by X, then a model is a system of equations describing the relationships among all variables, endogenous and exogenous. A "solution" of a model typically takes the form $Y^* = G(X)$. For example, the solution to Crusoe's problem is an allocation $Y^* = [\mathbf{c}^*, \mathbf{l}^*]$ which presumably depends on the particular values of $X = [\mathbf{A}, \boldsymbol{\lambda}, \beta, T]$.

As mentioned above, one of the most contentious issues in macroeconomics is the role of demand versus supply factors in explaining economic fluctuations. In Crusoe's island, the supply and demand forces are represented by the exogenous variables $\mathbf{A}$ and $\boldsymbol{\lambda}$ respectively. Therefore, a key task of our analysis is to assess the extent to which $\mathbf{A}$ and $\boldsymbol{\lambda}$ affect $\mathbf{c}^*$ and $\mathbf{l}^*$.

2.4 The Optimal Allocation

An optimal allocation is a solution to problem (2.4). A first observation is that resource constraints, $A_t l_t \geq c_t$, must hold with equality for allocations

[2]Mathematical issues about the existence and uniqueness of the solution are not fundamental at this stage and therefore left for the Appendix.

to be optimal. Otherwise a better allocation could be obtained by reducing labor and/or increasing consumption until the resource constraints binds.

Problem (2.4) can be simplified by eliminating c_t from the utility function using the resource constraints. The modified problem is:

$$\max_{\{l_t\}_{t=0}^T} \sum_{t=0}^{T} \beta^t \left[\lambda_t \ln \left(\underbrace{A_t l_t}_{c_t} \right) + \ln(1-l_t) \right].$$

This is a simpler problem than ((2.4)) because welfare depends only on $\mathbf{l}$ rather than $\mathbf{l}$ and $\mathbf{c}$. Let

$$W(\mathbf{l}) \equiv \sum_{t=0}^{T} \beta^t \left[\lambda_t \ln(A_t l_t) + \ln(1-l_t) \right]$$

be the function to be maximized. The dependence of W on $\mathbf{l}$ is made explicit by writing $W(\mathbf{l})$ meaning that $\mathbf{l}$ is what Crusoe chooses. W also depends on other parameters, namely β, T, $\mathbf{A}$ and $\boldsymbol{\gamma}$, which are beyond Crusoe's control. For l_0^* to be an interior maximizer of W then it must satisfy the condition $\frac{\partial W(\mathbf{l}^*)}{\partial l_0} = 0$. Using the definition of W, it follows that $\frac{\partial W(\mathbf{l}^*)}{\partial l_0} = \beta^0 \lambda_0 \frac{A_0}{A_0 l_0^*} - \beta^0 \frac{1}{1-l_0^*} = 0$. Similarly, the condition for l_1^* is $\frac{\partial W(\mathbf{l}^*)}{\partial l_1} = \beta^1 \lambda_1 \frac{A_1}{A_1 l_1^*} - \beta^1 \frac{1}{1-l_1^*} = 0$. More generally, the condition for l_t^* for $t = 0, 1, .., T$ can be can be written as:

$$\underbrace{\beta^t \lambda_t \frac{A_t}{A_t l_t^*}}_{\text{Marginal benefit of working}} = \underbrace{\beta^t \frac{1}{1-l_t^*}}_{\text{Marginal cost of working}} \quad \text{for } t = 0, 1, .., T. \tag{2.5}$$

Equation (2.5) is a rule describing the optimal labor choice. According to this condition, the optimal labor choice, l_t^*, is such that the marginal benefit of working an additional hour equals its marginal cost. Specifically, an extra hour of work produces A_t additional pounds of fish, and each pound of fish has a marginal utility of $\frac{\partial V}{\partial c_t^*} = \beta^t \lambda_t \frac{1}{c_t^*} = \beta^t \lambda_t \frac{1}{A_t l_t^*}$ utils. Therefore, $\beta^t \lambda_t \frac{A_t}{A_t l_t^*}$ are the additional "utils" of working one more hour. On the other hand, an additional hour of work reduces welfare by $-\frac{\partial V}{\partial l_t^*} = \beta^t \frac{1}{1-l_t^*}$ utils. Alternatively, the condition can be written as

$$A_t = -\frac{\partial V / \partial l_t^*}{\partial V / \partial c_t^*} = -\frac{\partial c_t}{\partial l_t^*} = \frac{c_t^* / \lambda_t}{1-l_t^*} = \frac{A_t l_t^* / \lambda_t}{1-l_t^*}$$

where the left hand side is the marginal rate of transformation and the marginal rate of substitution.

For each t, (2.5) provides one equation in one unknown, l_t^*. Solving for l_t^* produces $l_t^* = l(\lambda_t) = \frac{\lambda_t}{\lambda_t+1}$. The total production of the economy is $y_t^* = A_t l_t^*$, which also equals the total consumption as the resource constraint (2.3) holds with equality.

It is often convenient to organize the main results of an analysis in the form of propositions. A well stated proposition helps because it highlights the key results of the analysis.

Proposition. The optimal allocation of labor, consumption, and production are given by $l_t^* = l(\lambda_t) = \frac{\lambda_t}{\lambda_t+1}$ and $c_t^* = y_t^* = y(A_t, \lambda_t) = A_t \frac{\lambda_t}{\lambda_t+1}$ for $t = 1, .., T$.

The result stated in Proposition 2.4 is important because it provides the optimal solution, $(\mathbf{y}^*, \mathbf{l}^*)$, in terms of exogenous variables $(\mathbf{A}, \boldsymbol{\lambda})$. The proposition illustrates what a theory is all about: predictions about endogenous variables, in this case $(\mathbf{y}^*, \mathbf{l}^*)$, in terms of exogenous variables, in this case $(\mathbf{A},\boldsymbol{\lambda})$.

Notice that the optimal amount of labor effort, l_t^*, only depends on the preference parameter λ_t but not on the other parameters of the problem, namely $\mathbf{A}$, T, β or other λ different from λ_t. In particular, the more valuable consumption is at time t, as measured by λ_t, the higher the number of hours worked, a result that is intuitively clear. A perhaps unexpected result is that l_t^* is not affected by the productivity of labor, A_t. After all, a larger productivity of labor A makes fishing less costly which should induce Crusoe to consume more fish, what is known as *the substitution effect* in microeconomics.

However, a change in A also has an *income or wealth effect:* higher labor productivity effectively makes Crusoe richer because he can afford more consumption even without working more. We also know from microeconomics that the demand for normal goods increase when income increases. Therefore, if fish and leisure are normal goods, an increase in A should increase the demand for both consumption and leisure.

The previous considerations suggest that l_t^* may increase or decrease with A depending on the relative strength of the income and substitution effects. What effect dominates depends on the specific formulation of the utility function. It turns out that income and substitutions effects cancel each other out when $u_t(c_t, l_t)$ is of the log type, as in (2.2). As a result, the optimal labor choice, l_t^*, is independent of A_t. Section 2.8 considers an alternative formulation in which substitution and income effects do not cancel each other out.

It is significant that other parameters such as T or β do not affect the optimal allocation. The reason is that, in spite of its appearance, the problem is in fact static. Crusoe would like to exchange resources across periods, say by storing fish when A_t is high and consume it when A_t is low. However, this intertemporal reallocation of fish is unfeasible due to the perishable nature of fish. As a result, Crusoe treats the problem in each period as a separate problem. Parameters T and β would become relevant shortly once we enrich the model.

2.5 Data

We now explore some of the quantitative predictions of the model and its ability to replicate U.S. data. Quantitative exercises are a way to uncover strengths and weaknesses of the theories and to shed light on where to go next. It is natural to suspect that such a simple and unrealistic model would fail to explain any relevant feature of the U.S. economy. It turns out, however, that it offers some key and surprisingly robust insights. Furthermore, the model is useful because it allows us to illustrate techniques utilized by DSGE practitioners to connect models to reality.

The model offers predictions on production, consumption and labor decisions. Moreover, all production in the model is for consumption purposes. Therefore the natural starting place is to look for aggregate evidence on labor and consumption. Figure 2.1 and Figure 2.2 shows the evolution of labor hours and the share of employment in total population for the period Jan-64 to Nov-13. The raw information is from Federal Reserve Economic Database (FRED) and the time period is determined by the availability of information on hours per-week.

The workweek of the typical worker fell gradually during the period from 38.2 to 33.7 hours per week. In contrast, the fraction of the population at work increased more or less systematically, from 36% to 45%. This last trend reflects a substantial increase in female labor participation and aging of the population, that is, increase in the fraction of working age population. As a result, and in spite of the substantial reduction in the average workweek of a

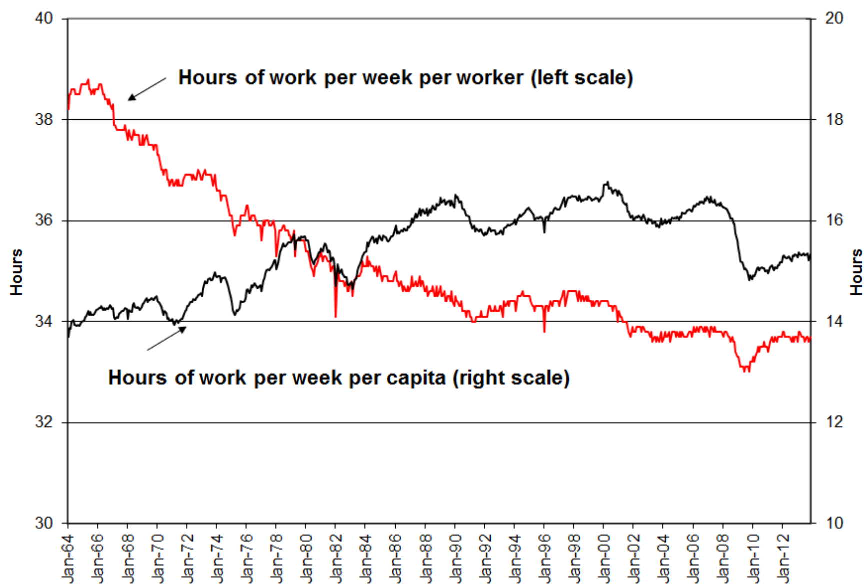

Figure 2.1: Hours of work

worker, the average person increased its workweek from 13.69 hours to 15.3 hours.

The opposite evolution of labor hours for workers, on one hand, and of population on the other, presents the first dilemma when using the model to explain data. In the model, all population (Crusoe) works. Should we use the model to explain labor hours of workers or labor hours of the total population? Since macro is interested in the overall performance of the economy, it seem more relevant to focus on total population. This will force us to address, rather than abstract from, key transformations of the post-war period such as the increase in female labor participation and aging. Taking this road requires us to re-interpret Crusoe's model as one of a family, or a *household,* composed, say, by a wife, a husband and a child, and reinterpret l_t and c_t as average labor supply and average consumption of the family respectively.

Figure 2.3 shows two series of real consumption in the US according to the National Income and Product Accounts (NIPA): consumption per-capita and consumption per-capita of non-durable goods, both relative to their value in January 1964. During the almost 50 year period, non-durable per-capita consumption grew by a factor of 2.13 while per-capita consumption grew

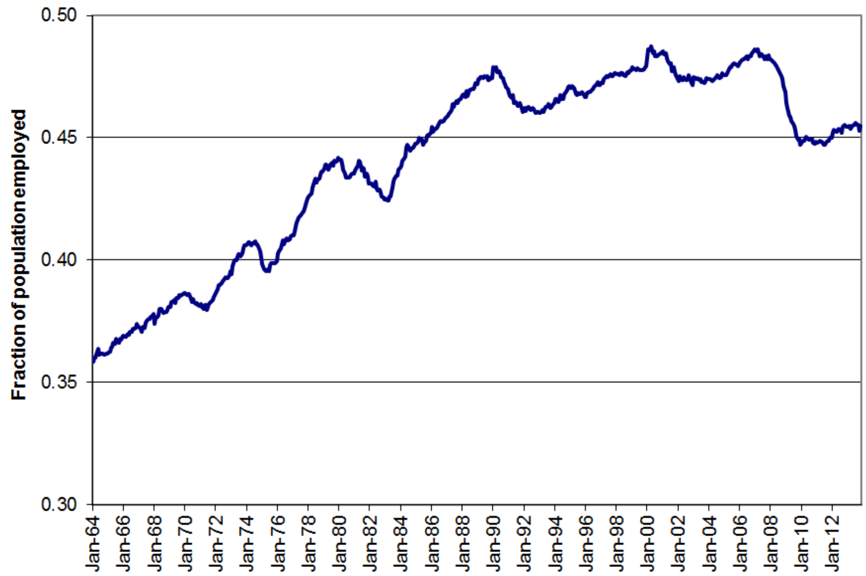

Figure 2.2: Fraction of the population employed

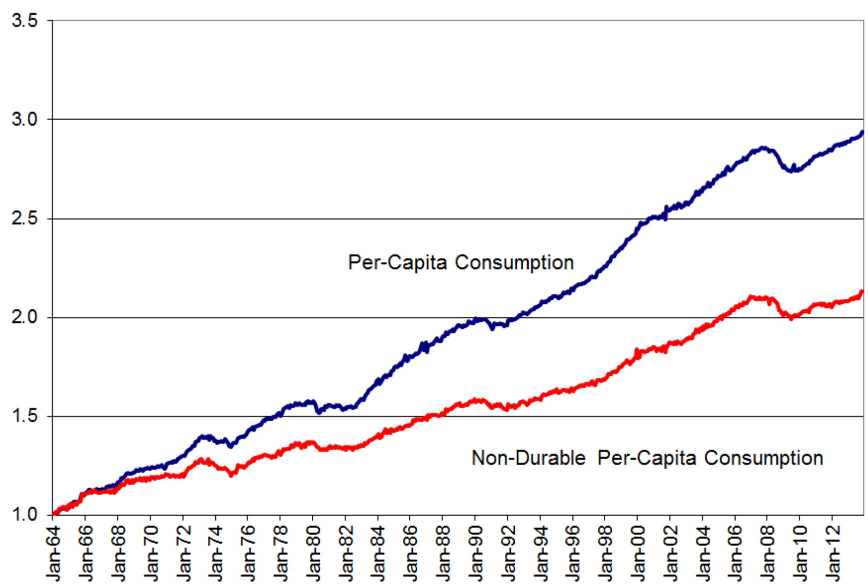

Figure 2.3: Consumption and durable consumption (per-capita).

by a factor of 2.94. This upward trend of per-capita consumption, or more generally, of total per-capita production, is known as *economic growth*. It means that individuals are becoming systematically richer, able to enjoy more consumption goods and services over time. This pattern of systematic, long run, improvement in living standards is relatively recent in human history and the most important regularity to be explained by macroeconomics.

A second feature of the data is the presence of short term economic fluctuations, or *business cycles*. As can be seen in Figure 2.3, the upward trend of consumption is not smooth but bumpy: sometimes the trend is more pronounced and sometimes it flattens. The most common way to characterize economic fluctuations in the news is by looking at the evolution of the growth rates of a variable[3]. Since we are using monthly data, we could look at monthly growth rates but it is more common to use annual growth rates defined as $g_t^X = \frac{X_t}{X_{t-12}} - 1$ where X refers to a particular variable such as consumption, labor or productivity. Since $g \simeq \ln(1+g)$ when g is a small number[4], I will use the following definition of a growth rate because it has better properties, as seen below:

$$g_t^X = \ln\left(\frac{X_t}{X_{t-12}}\right). \tag{2.6}$$

Figure 2.4 shows the annual growth rate of consumption per-capita. Growth fluctuates significantly and was negative during certain periods in 1974, 1980, 1990, 2001 and 2008-2009. Periods of persistent negative growth are called economic *recessions*. Periods of persistent significant growth are called *booms*. The average growth rate during the period was 2.14% per year. It is perhaps striking that such a small growth rate per year can produce such a large increase in consumption in a 50 year period.

Economic fluctuations, as measured by changes in g_t^X, can be characterized in terms of its range, average duration, average deviation, etc. I will focus on standard deviations and correlations. The standard deviation measures the average deviation of a variable from its mean. It can be easily computed in Excel using the function $stdev(X)$. The standard deviations of g_t^c is 1.93% in our data set. Intuitively, in a typical month the growth rate

[3] DSGE practicioners commonly use percentage deviations from trend. The trend is usualy obtained using the HP filter. I introduce the HP filter in Chapter 3 and discuss its advantages and disadvantages. Focusing on growth rates is still a relevant way to start.

[4] Use a first order Taylor approximation around $g = 0$ to confirm this result.

of consumption was either 1.93 percentage points above or below the mean of 2.14%.

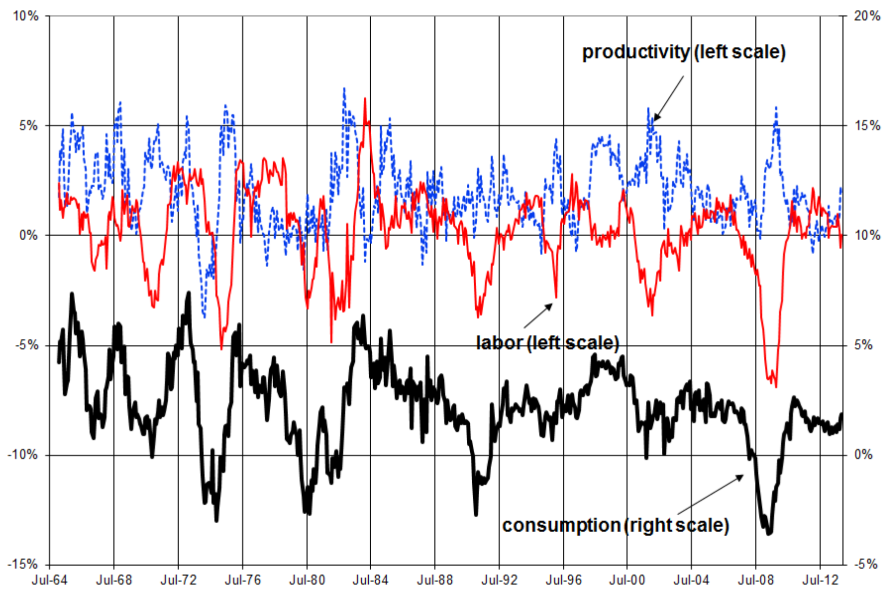

Figure 2.4: Annual Growth Rates of Per-Capita Consumption, Per-Capita Labor Hours, and Productivity

Figure 2.4 also displays the annual growth rates of per-capita labor hours and of labor productivity defined as $A_t = y_t/l_t$. The Figure suggests that recessions are typically associated to negative growth of labor hours, as in the 2008-2009 recession, and sometimes negative productivity growth, as in the 1974 recession.

A simple statistic that measures the degree of association, or commovement, between two variables is the *coefficient of correlation*, $\rho_{X,Y}$. The correlation is a number between -1 and $+1$ measuring the strength and direction of a linear relationship between two variables. A coefficient of -1 means a perfectly negative relationship, a coefficient of $+1$ means a perfectly positive relationship, and a coefficient of 0 indicates no relationship. This coefficient can be easily computed using the Excel formula $correl(X, Y)$. In our data set, the correlation between g^c and g^l is around 0.66 and the one between g^c and g^A is around 0.38. These correlations show an important but far from perfect association, one that is particularly strong between labor and consumption.

2.6 Calibrating the Model

So far we have described perhaps the simples microfounded macroeconomic model that offers predictions for total production, consumption and labor, and have reviewed basic aggregate data on consumption and labor. The next step is to connect the model to the data. In order to utilize the model and assess its performance it is necessary to specify the vectors of exogenous variables, $\mathbf{A}$ and $\boldsymbol{\lambda}$. This can be accomplished in different ways such as: (i) using evidence from microeconomics studies of firms and households regarding productivity and preferences; (ii) formal estimation of the model using panel data; or (iii) calibration. The exact methodology to connect models to data is by no means irrelevant nor unquestionable. In fact, part of the current major controversies in macroeconomics have to do with the empirical methods utilized[5]. Due to its simplicity and analytical clarity, this book emphasizes the last technique, calibration, but we discuss other approaches in later chapters.

Calibration is a technique that estimates the underlying exogenous variables by requiring the model to match certain features of the data known as targets. Typically one needs as many targets as exogenous variables. Once the exogenous variables are obtained, the model is then simulated and its performance assessed along dimensions beyond the matching targets. To illustrate this procedure, let's assume that the model is correct in its predictions of consumption and labor. This means that $c_t^* = c_t^{data}$ and $l_t^* = l_t^{data}$ for all t. Under these assumptions one can utilize Proposition 2.4 not for solving $\mathbf{c}^*$ and $\mathbf{l}^*$ but instead for solving $\mathbf{A}$ and $\boldsymbol{\lambda}$ as a function of the data, $\mathbf{c}^{data}$ and $\mathbf{l}^{data}$. Simple calculations results in:

$$A_t^e = \frac{c_t^*}{l_t^*} = \frac{c_t^{data}}{l_t^{data}} \text{ and } \lambda_t^e = \frac{l_t^*}{1-l_t^*} = \frac{l_t^{data}}{1-l_t^{data}}, \tag{2.7}$$

[5]See for example Edward C. Prescott, "Theory Ahead of Business Cycle Measurement," Fed. Res. Bank of. Min. Quarterly Review, Vol 10 (4), Fall 1986, pp. 9-22; Lawrence H. Summers, "Some Skeptical Observations on Real Business Cycle Theory", Fed. Res. Bank Minn. Rev., Vol 10(4), Fall 1986, pp. 23-27; Danny T. Quah, "Business Cycle Empirics: Calibration and Estimation," *Economic Journal*, Vol. 105, No. 433, Nov. 1995, pp. 1594-1596; Lars Peter Hansen and James J. Heckman, "The Empirical Foundations of Calibration," *Journal of Economic Perspectives*, Vol. 10, No. 1, Winter, 1996, pp. 87-104; Ramdam Dridi, Alain Guaya, and Eric Renaultb, "Indirect Inference and Calibration of Dynamic Stochastic General Equilibrium Models," *Journal of Econometrics,* Vol. 136 (2), February 2007, pp. 397-430.

where superscript "e" stands for "estimated".

In order to utilize (2.7), we need to define c_t^{data} and l_t^{data}. Since a key assumption of the model is that goods are non-storable, it is natural to define c_t^{data} as non-durable consumption. On the other hand, we would need labor only employed in the production of non-durable consumption, but this classification of labor is harder to find. For purposes of illustration, I define c_t^{data} as per-capita consumption and l_t^{data} as per-capita labor hours per-week. This choice makes the results below more suggestive than anything, a first pass. As for hours, I divide hours by total hours available per-week so that l_t^{data} is a number between 0 and 1 as required by the model.

Figure 2.5 displays the estimated parameters, and the calculations can be found in the companion Excel file *crusoe.xls* (see worksheet *data*). We discuss below some implications of these estimates, but before taking them too seriously, the next step would be to assess the predictions of the model in dimensions beyond the matching targets, that is, beyond consumption and labor. Such assessment is crucial because it would give us some idea about how useful the theory at hand is in explaining the evidence. Unfortunately, we have not derived further predictions of the theory yet. This task is left for the next chapter where predictions about wages and interest rates are considered. For the remaining of the chapter we proceed as if the model in fact provides an useful description of the US economy. In this sense, the exercises that follow is really an *accounting exercise* because the untested theory is used as a framework to organize ideas.

2.7 Results

Calibration exercises may be extremely informative on their own. If a theory is particularly appealing for some reason, say because it builds on compelling assumptions, then it can be used to estimate the underlying exogenous variables and parameters required to replicate the observed behavior. The results shed light on the fundamental driving forces of the economy[6]. In our case,

[6] An example is Ana Castañeda, Javier Diaz Gimenes, and Jose-Victor Rios-Rull, "Accounting for the U.S Wealth Inequality," Journal of Political Economy, Vol. 111 (August 2003). They back up the underlying driving forces generating wealth inequality in the U.S. using a particular well-regarded theory. Another example is V.V. Chari, Patrick Kehoe and Ellen McGratan, "Business Cycle Accounting," Econometrica, Vol. 75(3) (2007), pp. 781-836 . They use basic theory to back up fundamental wedges.

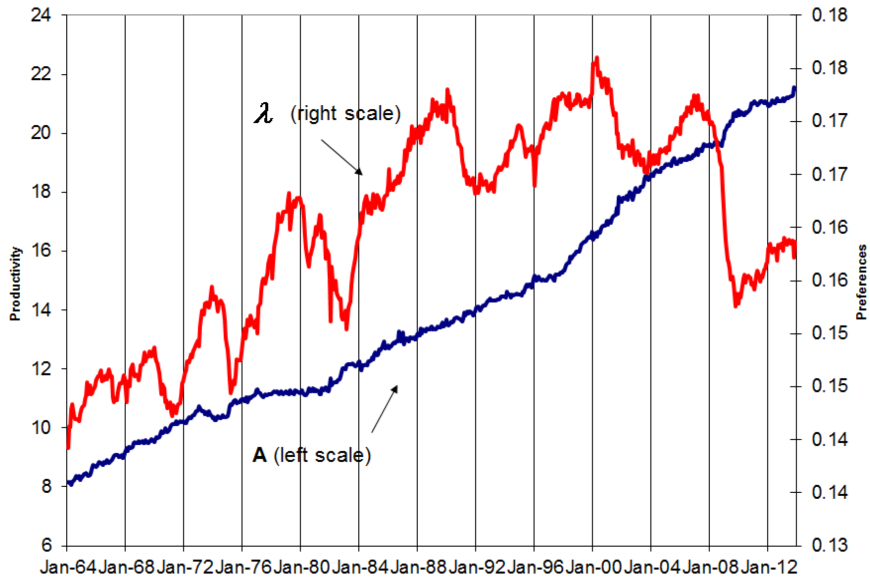

Figure 2.5: Calibrated Values of $\boldsymbol{\lambda}$ and **A**.

the estimates of $\{A_t\}_{t=0}^{T}$ and $\{\lambda_t\}_{t=0}^{T}$ can be used to assess, at the light of the model, the importance of demand versus supply forces in determining economic growth and business cycles.

2.7.1 Economic Growth

Figure 2.5 reveals several interesting results. The first one is the systematic and substantial increase in the productivity of labor, A_t^e, which multiplied by a factor of 2.63 during the period. Since per-consumption, $c_t = A_t l_t$, multiplied by a factor of 2.94 during the same period, then most of the increase in consumption was due to productivity gains. Thus, in order to understand the origin of economic growth it seems key to understand the origin of the labor productivity gains. Since A_t is exogenous, the model of this section is completely silent about the determinants of labor productivity but we address this important issue in later chapters.

A second finding is the (weak) positive trend of λ_t^e, abruptly interrupted by the great recession of 2008, meaning that the value of consumption, relative to leisure, has increased over time. Notice that the computation of λ_t^e, given by (2.7), relies on per-capita labor hours, l_t^{data}, and in particular, the larger l_t^{data} the larger λ_t^e. This means that the model explains the increase in female labor participation and aging as the result of increase in λ_t, that is, an increase in the desire for consumption goods, at the expense of leisure. Is this a satisfactory explanation for the trend in labor hours? It depends. Regarding aging, increasing λ_t may be a simple and sensible way to capture this demographic trend, although one can only be sure if aging is explicitly included into the model. As for the increase in female labor participation, there are multiplicity of explanations for this trend, ranging from cultural, political, technological, and/or demographic changes[7]. Describing all these forces by a simple variable $\boldsymbol{\lambda}$ may be an oversimplification. However, this is not necessarily a weakness of the model and could instead be its strength. One can interpret the substantive role of $\boldsymbol{\lambda}$ as one of encapsulating all other major forces shaping the economy besides productivity, $\mathbf{A}$. The model would thus provide a clean division of the fundamentals driving forces of the economy into technological factors, as represented by $\mathbf{A}$, and other factors, as represented by $\boldsymbol{\lambda}$. This division may be referred to as supply versus demand factors. Now, the only way to gain or lose confidence in the model is to put it to compete with other models, a horse race, and see how it performs: "it takes a model to beat a model." We do this as we move along.

Figure 2.6 shows consumption per capita and two *counterfactual* series of consumption, c_t^λ and c_t^A. These series describe the separate effect of only one fundamental variable at a time, either λ or A respectively. Specifically, for $t = 0, 1, .., T$:

$$c_t^\lambda = y\left(A_0^e, \lambda_t^e\right) = A_0^e \frac{\lambda_t^e}{1+\lambda_t^e} \text{ and } c_t^A = y\left(A_t^e, \lambda_0^e\right) = A_t^e \frac{\lambda_0^e}{1+\lambda_0^e}. \tag{2.8}$$

The calculations can be found in the companion Excel file *crusoe.xls*. It is clear from Figure 2.6 that A is the most important determinant of the upward trend of consumption.

What fraction of long term growth, or just growth, is explained by $\mathbf{A}$ and $\boldsymbol{\lambda}$ respectively? These fractions, or *contributions,* could be computed by as $\phi_g^A = \frac{g^{c^A}}{g^c}$ and $\phi_g^\lambda = \frac{g^{c^\lambda}}{g^c}$ where $g^x = \frac{x_T}{x_0} - 1$ is the growth rate of variable x during the whole period and $x = \left\{c, c^A, c^\lambda\right\}$. Thus, ϕ_g^A is the fraction of

[7] See for example Greenwood, Seshadri, and Yorokoglu (2005), Engines of Liberation, Review of Economic Studies, 72(1): 109-33., and Jones, Manuelli, and McGrattan (2003), Why Are Married Women Working So Much?, Federal Reserve Bank of Minneapolis, Staff Report: 317.

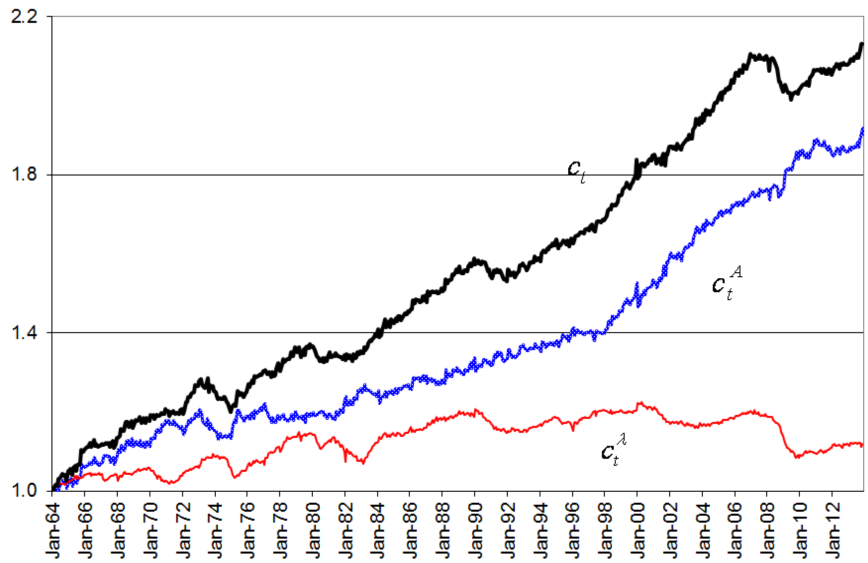

Figure 2.6: Per-capita Consumption and Its Determinants

growth due to productivity and ϕ_g^λ is the fraction of growth due to other factors. These ratios typically would work fine but more precise formulas can be obtained as follows. Notice first that $c_t = \frac{c_t^A c_t^\lambda}{c_0}$ and therefore,[8]

$$\ln\left(1+g_t^c\right) = \ln\left(1+g_t^{c^A}\right) + \ln\left(1+g_t^{c^\lambda}\right). \tag{2.9}$$

That is the growth rate of consumption can be represented in terms of the growth rates of c_t^A and c_t^λ. As a result,

$$1 = \frac{\ln\left(1+g_t^{c^A}\right)}{\ln\left(1+g_t^c\right)} + \frac{\ln\left(1+g_t^{c^\lambda}\right)}{\ln\left(1+g_t^c\right)}.$$

[8]In particular,

$$\begin{aligned} 1+g^c &= \frac{c_T}{c_0} = \frac{A_T}{A_0}\frac{l\left(\lambda_T\right)}{l\left(\lambda_0\right)} = \frac{A_T l\left(\lambda_0\right)}{A_0 l\left(\lambda_0\right)}\frac{A_0 l\left(\lambda_T\right)}{A_0 l\left(\lambda_0\right)} = \frac{c^A\left(A_T,\lambda_0\right)}{c^A\left(A_0,\lambda_0\right)}\frac{c^\lambda\left(A_0,\lambda_T\right)}{c^\lambda\left(A_0,\lambda_o\right)} = \frac{c_T^A}{c_0^A}\frac{c_T^\lambda}{c_0^\lambda} \\ &= \left(1+g^{c^A}\right)\left(1+g^{c^\lambda}\right). \end{aligned}$$

This expression suggest the following definition for the contribution of A and λ to economic growth:

$$\phi_g^A = \ln\left(1+g^{c^A}\right)/\ln\left(1+g^c\right) \text{ and } \phi_g^\lambda = \ln\left(1+g^{c^\lambda}\right)/\ln\left(1+g^c\right). \quad (2.10)$$

Since $\ln(1+a) \simeq a$ when a is a small number, then defining ϕ_g^A and ϕ_g^λ one way or the other does not makes much difference when growth rates are small. Some problems arise when growth rates are large. In that case the second formulation is more precise. For this reason, we would use the second formulation. Applying these formulas to our problem produces $\phi_g^A = 0.89$ and $\phi_g^\lambda = 0.11$. This confirms the visual impression of Figure (2.6) that most growth during the last 50 years was due to productivity gains.

2.7.2 Business Cycles

A fundamental question in macro is the origin of economic fluctuations. Do fluctuations originate mostly on demand or supply factors? Figure 2.6 shows that productivity plays the dominant role in explaining overall growth but it also suggests that other forces captured by $\boldsymbol{\lambda}$ may play a significant role in explaining short term *fluctuations*. To see this more clearly, notice from the definitions of c_t^A and c_t^λ in (2.8) that the annual growth rate of c_t^A is the annual growth rate of productivity, g_t^A, while the annual growth rate of c_t^λ is the annual growth rate of labor, g_t^l, both of them shown in Figure (2.4). The figure shows an strong association between annual consumption growth, g_t^c, and annual labor growth, and therefore suggests that $\boldsymbol{\lambda}$ is key in generating consumption fluctuations.

A more formal way to assess the contribution of A and $\boldsymbol{\lambda}$ to short term fluctuations is to compute ratios similar to that in (2.10) but for each month. This approach would quickly run into troubles because in some particular months g_t^c would be zero or very close to zero creating unrealistic very large ratios. An alternative that avoids these problems defines the contributions to fluctuations using variance decomposition (as in Klenow and Rodriguez-Clare 1997). The additive law of covariance states that $cov(x+y,z) = cov(x,z) + cov(y,z)$, where $covar(x,y)$ stands for covariance between x and y, and $var(x)$ stands for variance of x. Thererefore, using (2.9):

$$\begin{aligned} var\left(\ln\left(1+g^c\right)\right) &= cov\left(\ln\left(1+g^c\right), \ln\left(1+g^c\right)\right) \\ &= cov\left(\ln\left(1+g^c\right), \ln\left(1+g^{c^A}\right)\right) + cov\left(\ln\left(1+g^c\right), \ln\left(1+g^{c^\lambda}\right)\right). \end{aligned}$$

or

$$1 = \varphi_f^A + \varphi_f^\lambda$$

where

$$\varphi_f^A = \frac{covar(\ln\left(1 + g^{cA}\right), \ln(1 + g^c))}{var(\ln(1 + g_t^c))} \text{ and } \varphi_f^\lambda = \frac{covar(\ln\left(1 + g_t^{c\lambda}\right), \ln(1 + g_t^c))}{var(\ln(1 + g_t^c))}. \tag{2.11}$$

Intuitively, φ_f^A measure the fraction of the variation of g_t^c that is explained by variation of g^{cA}, and similarly for φ_f^λ.[9] As before, when growth rates are small one can approximate φ_f^λ and φ_f^A by

$$\varphi_f^A \simeq \frac{covar(g_t^{cA}, g_t^c)}{var(g_t^c)} \text{ and } \varphi_f^\lambda \simeq \frac{covar(g_t^{c\lambda}, g_t^c)}{var(g_t^c)}.$$

These formulas can be easily computed using the functions $covar()$ and $var()$ of Excel. One finds $\varphi_f^A = 0.32$ and $\varphi_f^\lambda = 0.68$. These results confirm the visual impression of Figure (2.4) that fluctuations in λ account for most of the business cycles but fluctuations of productivity are also important.

2.8 Extensions

Although models are silent about the determinants of exogenous variables, one can use models to learn about them, as we did in the previous section. We found that both $\mathbf{A}$ and $\boldsymbol{\lambda}$ have an upward trend which can be gauged by their strong correlation of 0.59 ($= correl(\mathbf{A}, \boldsymbol{\lambda})$). Is such a strong correlation between exogenous variables problematic? It may. One can argue, for example, that the upward trend in per-capita labor hours is actually explained by productivity gains as they make labor markets more attractive for individuals. If this explanation is correct then the model above is the wrong model to look at because optimal labor, $\mathbf{l}^*$, does not respond to $\mathbf{A}$. We now consider an extension of the model that delivers the required prediction.

[9] As explained by Klenow and Rodriguez-Clare (1997, pg 8), the decomposition is equivalent to looking at the coefficients from independently regressions of $ln(1 + g^c)$ against $ln(1 + g^A)$ and $ln(1 + g^\lambda)$, respectively. The coefficients sums to one since OLS is a linear operator and $ln(1 + g^c) = ln(1 + g^A) + ln(1 + g^\lambda)$. The decomposition amounts to asking, "When we see 1% higher $(1 + g^c)$ in one period relative to the mean, how much higher is our conditional expectation of g^A and how much higher is our conditional expectation of g^λ?"

2.8.1 A Generalized Model

Consider the economy of the previous section but suppose that instead of (2.2) Crusoe's preferences are now described by:

$$u_t\left(c_t, 1-l_t\right) = \frac{1}{\theta}\left(\lambda_t c_t^{\theta} + \left(1-l_t\right)^{\theta}\right) \text{ where } \theta < 1. \tag{2.12}$$

Here θ is a parameter assumed to be known. The restriction $\theta < 1$ is required for a well-defined concave problem. Other aspects of the model are the same, and in particular $c_t = A_t l_t$. Following the same considerations of the previous section, Crusoe's problem can be described as:

$$\max_{l_0, l_1, l_2, .., l_T} \sum_{t=0}^{T} \beta^t \frac{1}{\theta}\left(\lambda_t \left(\underbrace{A_t l_t}_{c_t}\right)^{\theta} + \left(1-l_t\right)^{\theta}\right).$$

The first order conditions for a maximum are:

$$\beta^t \lambda_t A_t^{\theta} l_t^{*\theta-1} - \beta^t \left(1-l_t^*\right)^{\theta-1} = 0 \text{ for } t = 0, 1, .., T. \tag{2.13}$$

For each t, (2.13) is one equation in one unknown, l_t^*. Solving for l_t^* from this equation produces, after some manipulations[10]:

$$l_t^* = l(\lambda_t, A_t) = \frac{1}{1 + \lambda_t^{\frac{1}{\theta-1}} A_t^{\frac{\theta}{\theta-1}}}. \tag{2.14}$$

Furthermore, $y_t^* = A_t l_t^*$ is given by:

$$y_t^* = y(A_t, \lambda_t) = \frac{A_t}{1 + \lambda_t^{\frac{1}{\theta-1}} A_t^{\frac{\theta}{\theta-1}}}. \tag{2.15}$$

Notice that when $\theta = 0$ we obtain the same solutions of the previous section, and in particular l_t^* does not depend on A_t in that case. However, if $\theta > 0$ then l_t^* and A_t are positively related while if $\theta < 0$ the relation is reversed. To see this, notice that $\theta - 1$ is always negative and, therefore $A_t^{\frac{\theta}{\theta-1}}$ decreases

[10] Rewrite (2.13) as $\lambda_t A_t^{\theta} l_t^{*\theta-1} = \left(1-l_t^*\right)^{\theta-1}$. Raise both sides of this equation to the power $\frac{1}{\theta-1}$. This produces $\lambda_t^{\frac{1}{\theta-1}} A_t^{\frac{\theta}{\theta-1}} l_t^* = 1 - l_t^*$. Collect terms to get $\left(\lambda_t^{\frac{1}{\theta-1}} A_t^{\frac{\theta}{\theta-1}} + 1\right) l_t^* = 1$. Finally, solving for l_t^* gives the desired result.

(increases) with A if $\theta > 0$ ($\theta < 0$). According to (2.14), this means that l_t^* increases (decreases) with A_t if $\theta > 0$ ($\theta < 0$)[11]. In conclusion, labor hours may have increased due to productivity gains if θ is positive.

The model can be used to assess the role of $\mathbf{A}$ and $\boldsymbol{\lambda}$ for growth and fluctuations. This requires to specify the exogenous variables $\mathbf{A}$, $\boldsymbol{\lambda}$ and θ. The values calibrated in the previous section cannot be used because they were found using assumptions specific to the model of that section. A new model requires a new calibration.

As in the previous section, assume that $c_t^* = c_t^{data}$ and $l_t^* = l_t^{data}$. Since the model satisfies $c_t^* = A_t l_t^*$, the calibration of A_t is the same: $A_t^e = \frac{c_t^{data}}{l_t^{data}}$. Knowing A_t^e, one can use (2.13) to solve for λ_t as:

$$\lambda_t^e = (A_t^e)^{-\theta}\left(\frac{1 - l_t^{data}}{l_t^{data}}\right)^{\theta - 1}. \tag{2.16}$$

In order to use this formula to find λ_t^e, a value of θ is needed.

Let's consider two alternative calibrations. The first one calibrates θ so that $correl(\boldsymbol{\lambda}^e, \mathbf{A}^e) = 0$ as discussed above. Although this *identification* procedure seems sound, it also has a seemingly problematic implication, as shown below: it produces a strong (negative) correlation between g_t^A and g_t^λ. This may be problematic as it suggests that business cycles may not be the result of two independent driving forces, g_t^A and g_t^λ, but instead of one single common force. To address this issue, we also explore an alternative calibration that identifies θ by requiring $correl(g_t^A, g_t^\lambda) = 0$.

2.8.2 Calibration 1

The first calibration of the generalized model picks θ so that $correl(\boldsymbol{\lambda}^e, \mathbf{A}^e) = 0$. Since $\mathbf{A}$ has a positive trend, this zero correlation would mean that $\boldsymbol{\lambda}$ has no trend, and therefore the observed systematic increase in labor hours must be explained by $\mathbf{A}$ only. This identification procedure is in the spirit of Blanchard and Quah (1989).

To solve for θ and λ_t^e in a computer use the following simple procedure: (i) pick an arbitrary value for θ, say $\theta = 0$;(ii) Compute λ_t^e for all t using

[11] Alternatively, one can compute $\frac{\partial l_t^*}{\partial A_t} = -\frac{\theta}{\theta-1}\frac{A_t^{\frac{\theta}{\theta-1}-1}}{\left(1+\lambda_t^{\frac{1}{\theta-1}}A_t^{\frac{\theta}{\theta-1}}\right)^2}$ which is positive if $\theta > 0$ and negative if $\theta < 0$.

(2.16); (iii) Compute $correl(\boldsymbol{\lambda}^e, \mathbf{A}^e)$; If this correlation is positive, increase θ, if it is negative reduce θ; Stop when $correl(\boldsymbol{\lambda}^e, \mathbf{A}^e)$ is zero (close to zero in practice). The procedure is implemented in the companion Excel file *crusoe.xls* (see worksheet *calculations*). After a minute or two of trial and error (by hand) one finds that $\theta = 0.118$ produces the required correlation. This positive value of θ means that the substitution effect dominates the income effect: as labor productivity increases households get richer but also leisure becomes more expensive. This last effect dominates and as a result households supply more labor.

Results

As in the previous section, one can construct the *counterfactual* series

$$c_t^{\lambda} = y\left(A_0^e, \lambda_t^e\right) \text{ and } c_t^A = y\left(A_t^e, \lambda_0^e\right) \text{ for } t = 0, 1, .., T,$$

where $y\left(A^e, \lambda^e\right)$ is defined in (2.15). Once the series are constructed, they can be used to assess the contribution of different fundamentals in explaining the data. For example, for this model one finds that $\varphi_g^A = 99.9\%$ while $\varphi_g^{\lambda} = 0.1\%$. This means that long term economic growth is fully accounted by technological progress. This is expected because by construction the trend in labor is accounted by changes in A. A more interesting result is that $\varphi_f^A \simeq 35.7\%$ and $\varphi_f^{\lambda} = 64.8\%$, which gives slighly but not significanlty more role to productivity. The central role of productivity in accounting for business cycles is the focus of *real business cycles theories*.

An issue with the model, however, is that large negative correlation between g_t^{λ} and g_t^A, of -51.2%. This is illustrated in Figure (2.7). Notice first that both g_t^{λ} and g_t^A are both positively associated with g_c, so that, according to the model, recessions are typically periods of both low productivity growth and low demand of consumption goods. In spite of this, g_t^{λ} and g_t^A turn out to be negatively associated. For example, according to the model, in 1968, 1975, 1982, 2002 and 2009 g_t^{λ} reached historical highs while g_t^A reached historical lows.

Part of the problem arises precisely because the θ required to explain the trend in labor hours also implies that labor hours must also respond strongly to changes of productivity in the short term. Thus, for example, in the 2009 recession g_t^A was high, as shown in Figure 2.7, and therefore labor hours should have been relatively high. However, as seen in Figure 2.4, labor hours

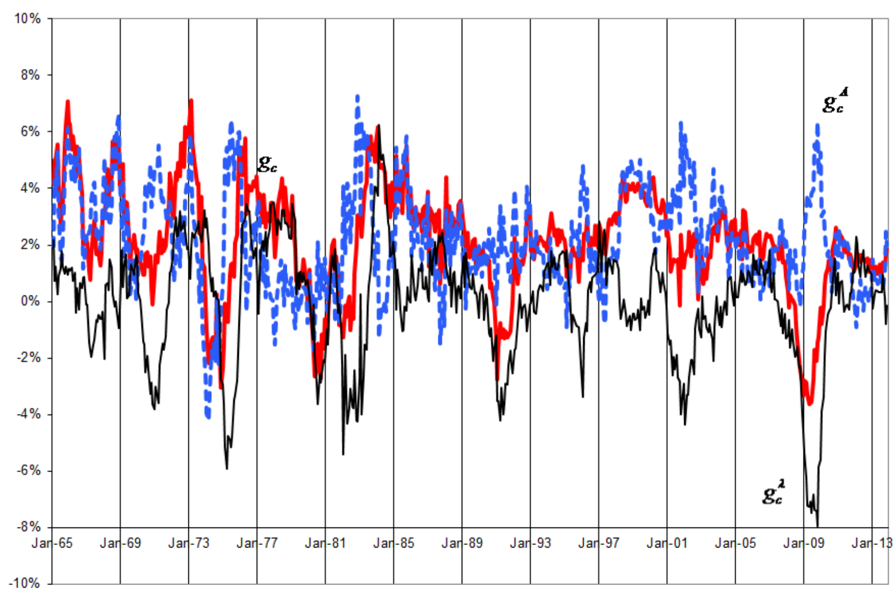

Figure 2.7: Growth rates of c, c^A and c$^\lambda$

were actually low. This can only be explained by the model if consumption had a particularly low value (λ_t was low). Thus, high productivity is associated to low λ. This is also what Figure 2.7 shows for the 2001 recession.

In summary, the model can account for the trend in labor hours as the result in productivity gains, but at the same time produces a strong negative correlation between exogenous variables at the business cycle frequency that may be problematic. This negative correlation suggests that there is only one force driving the business cycles, not two. We explore this issue next.

2.8.3 Calibration 2

An alternative way to identify θ is to require $correl(g^\lambda, g^A) = 0\%$. This is accomplished by setting $\theta = -1.530$. The negative value of θ obtained means that the income effect dominates the substitution effect. Thus, the relative high productivity during the 2001 recession would have reduced labor supply instead of increasing it. Regarding economic fluctuations one finds that $\varphi_f^\lambda \simeq 81\%.4$ and $\varphi_f^A = 15\%$. In this case, business cycles are more strongly influenced by fluctuations in preferences, or in the demand side, rather than in technologies, or in the supply side. This is the main argument

	θ		
	0.000	0.118	-1.530
	Contributions to Growth		
φ^A_g	89.5%	99.9%	40.5%
φ^λ_g	10.5%	0.1%	55.1%
	Standard Deviations		
σ_{gc}	1.93%	1.93%	1.93%
σ_{gcA}	1.61%	2.09%	1.71%
$\sigma_{gc\lambda}$	1.99%	1.80%	0.74%
	Correlations		
$\rho_{gc,gcA}$	38.2%	38.2%	92.0%
$\rho_{gc,gc\lambda}$	66.1%	59.8%	39.1%
$\rho_{gA,g\lambda}$	-44.1%	-51.2%	0.0%
$\rho_{A,\lambda}$	59.1%	0.0%	97.7%
	Contributions to Fluctuations		
φ^A_f	31.9%	35.7%	14.9%
φ^λ_f	67.9%	64.8%	81.4%

Figure 2.8: Some Statistics from Different Models

made by neo-keynesian theories.

A potential weakness of this approach is that now the trend in productivity and the fact that θ is negative, should have reduced labor supply in the long term due to the strong income effect. However, since actually labor increased during the period, then the model would require a systematic increase in λ that makes leisure less valuable. As a result now $correl(\lambda^e, A_t^e)$ is 97.8. Moreover, $\varphi_g^A = 40.5\%$ and $\varphi_g^\lambda = 55\%$. This means that, according to the model, 55% of the observed economic growth would not have happened without the large increase in the preference for consumption!

Table (2.8) summarizes some of the results presented in this chapter.

2.9 Social Planners and Efficient Allocations

Crusoe's problem is an example of a social planner's problem and $\{c_t^*, l_t^*\}_{t=0}^T$ an example of a social planner's allocation, also called an *efficient allocation* or a *Pareto allocation*. This is because Crusoe acts trivially as a *benevolent* dictator, or a social planner, who seeks to maximize the welfare of the society subject only to fundamental resource constraints. In particular, Crusoe is not subject to market prices such as wages or fish prices, as it will be the case in the next chapter. Facing only technological constraints, a social planner can obtain what is best for the society and therefore no other allocation is better than a social planner solution. If so, then the sort of economic fluctuations and economic growth rationalized by the model, as illustrated above, are socially optimal. No intervention is necessary.

Social planner allocations play an important role in macroeconomics. In particular, they serve as the benchmark against which to compare other allocations such as the one obtained by competitive markets. As we see in the next chapter, allocations obtained by competitive markets could be identical to the social planner allocation under certain conditions.

2.10 Conclusions

This chapter illustrates in a very simplified way what macroeconomists typically do, and the problems they face. They usually have in mind a theory of how the economy works, a model. In this chapter we started with a very particular theory, and look at the data from the point of view of this theory. We learned how to solve the model, and how to calibrate it. We found that the same model may give rise to very different insights depending on how key exogenous driving forces are identified. The issue of identification is a difficult one, at the heart of major controversies. A shortcoming of the chapter is that we didn't really test the models. We simply assumed they were right. We make some progress along this and other dimensions in the next chapter.

2.11 Exercises

1. Prove that
$$\lim_{\theta\to 0} = \frac{1}{\theta}\left(\lambda_t c_t^{\theta} + (1-l_t)^{\theta} - \lambda_t - 1\right) = \lambda_t \ln c_t + \ln(1-l_t).$$

2. Suppose total time available for work and leisure in Section 2.8 is h rather than 1. Solve the problem and compare the new solutions with those obtained in that section. Show that the solution for the problem with $h = 1$ can be used to find the solution for any h.

3. Suppose lifetime utility is described by $W = \theta V^{\frac{1}{\theta}}$ where V and u_t are defined by (2.1) and (2.12). Notice that W is a monotonic transformation of V. Show that all results of Section (2.8) remain intact.

4. Suppose the per-period utility function in the Crusoe's model is $u_t(c_t, l_t) = \frac{\left(c_t^{\lambda_t}(1-l_t)\right)^{\mu}}{\mu}$ where μ is a parameter. Find the optimal allocations, $\{c_t^*, l_t^*\}_{t=0}^{T}$ as a function of parameters, and compare with the results obtained for the case $u_t(c_t, l_t) = \lambda_t \ln c_t + \ln(1 - l_t)$. Can you give a reason for why the results are similar or different?

5. Suppose that a period is a day and the maximum amount of time available is 24 hours in the model of Section 2.8.

   (**a**) Find $\{c_t^*, l_t^*\}_{t=0}^{T}$ (as a function of parameters).

   (**b**) Calibrate λ so that Crusoe works 8 hours a day every period. Report the value obtained.

6. Calibrate Crusoe's model using per-capita non-durable consumption instead of consumption per-capita. Discuss the resulting A_t and λ_t by comparing your results to the ones obtained using per-capita consumption. Construct two well-labeled graphs, one for A_t and one for λ_t, comparing the calibrated values in both models. Discuss.

7. Suppose $u_t(c_t, l_t) = \lambda_t c_t + \ln(1 - l_t)$ in Crusoe's model. Notice that preferences are "quasilinear."

   (**a**) Find $\{c_t^*, l_t^*\}_{t=0}^{T}$?

   (**b**) Calibrate A_t and λ_t to match consumption per-capita and per-capita hours of work. Construct a well-labeled graph displaying the calibrated values of A_t and λ_t.

   (**c**) Calculate and discuss the contribution to growth and business cycles of supply and demand factors.

8. Crusoe's life time utility is described by:

$$\sum_{t=0}^{T} \beta^t \left(\lambda_t \ln c_t + \gamma_t (1 - l_t)\right)$$

where c_t is consumption and l_t is labor, and $\{\lambda_t, \gamma_t\}_{t=0}^{T}$ are exogenous variables. Crusoe produces $A_t l_t$ units of consumption good with l_t hours of work. He chooses $\{c_t, l_t\}_{t=0}^{T}$ in order to maximize his welfare subject to the technological restrictions $c_t \leq A_t l_t$ for $t = 0, 1, .., T$. Let $\{c_t^*, l_t^*\}_{t=0}^{T}$ be the optimal choices.

(**a**) Set up Crusoe's maximization problem and find $\{c_t^*, l_t^*\}_{t=0}^T$.

(**b**) Suppose you have data on consumption and labor: $\left\{l_t^{data}, c_t^{data}\right\}$. Describe how would you calibrate the underlying exogenous variables of the model.

(**c**) Denote $\{A_t^e, \lambda_t^e, \gamma_t^e\}_{t=0}^T$ the calibrated values. Suppose you compute the *counterfactual* series:

$$c_t^\gamma = A_0^e l\left(\alpha_t^e, \gamma_t^e\right) \text{ and } c_t^A = A_t^e l\left(\alpha_0^e, \gamma_0^e\right) \text{ for } t = 0, 1, .., T.$$

where $l\left(\lambda_t^e, \gamma_t^e\right) = l_t^*$ is the optimal labor choice (a function of the exogenous variables). Moreover, suppose you find that $\varphi_g^A = 75\%$ and $\theta_g^{\lambda,\gamma} = 25\%$. What do these numbers suggest?

(**d**) Suppose you find that $\varphi_f^\gamma = 15\%$ and $\varphi_f^A = 85\%$. What do these numbers suggest?

9. Consider the problem

$$\max_{l\in[0,1]} \sum_{t=0}^{T} \beta^t u(f(l), 1-l).$$

Find conditions for u and f under which the solution to this problem exists and is unique.